ABSTRACT OF SEMINAR: QUANTITATIVE STRATIFICATION AND APPLICATIONS TO HARMONIC FUNCTIONS

DANIELE VALTORTA, EPFL

The main objective of this talk is to present the quantitative stratification technique. In order to do so, we use the application to the critical sets of harmonic function as a guiding example.

The quantitative stratification is a technique based on the analyis of the symmetries of the object under study at different points in its domain, and their interactions with each other. It was introduced by prof. Cheeger and Naber in [CN13, CN12].

In detail, for harmonic functions we prove effective estimates on the volume of the tubular neighborhood of the critical set of $u : B_1(0) \to \mathbb{R}$ of the form:

$$\mathrm{Vol}(B_r(\mathcal{C}(u)) \cap B_{1/2}(0)) < C(n, \bar{N}^u(0,1), \epsilon) r^{2-\epsilon}\,, \tag{1}$$

where $\epsilon > 0$ is arbitrary, and $\bar{N}^u$ is Almgren's frequency function.

Given a harmonic function u, the standard *first order* stratification $\{\mathcal{S}^k\}$ of u separates points x based on the degrees of symmetry of the leading order polynomial of $u - u(x)$. In this talk we introduce a quantitative stratification $\{\mathcal{S}^k_{\eta,r}\}$ of u, which separates points based on the number of *almost* symmetries of *approximate* leading order polynomials of u at various scales.

By studying how nearby symmetric points interact, we prove an effective cone splitting theorem, and the monotonicity of the frequency function $\bar{N}^u(r)$ will allow us to bound the number of scales where a function is not close to be symmetric. With these tools available, we use a simple covering argument to prove volume estimates on the tubular neighborhoods of $S^k_{\eta,r}$. The estimates on the critical sets follow as a simple corollary.

Up to some technical detail, these techniques can be also adapted to prove estimates on the solutions to more general elliptic PDEs.

These results are described in the article [CNV]. Other results available on the subject are described in [HL, HHL98, HL00].

References

[CN12] J. Cheeger and A. Naber, Quantitative stratification and the regularity of harmonic maps and minimal currents, *Comm. Pure and Appl. Math.* (2012). Available at `http://arxiv.org/abs/1107.3097`.

[CN13] ———, Lower bounds on Ricci curvature and quantitative behavior of singular sets, *Invent. Math.* **191** (2013), 321–339. MR 3010378. Zbl 06143195. `http://dx.doi.org/10.1007/s00222-012-0394-3`. Available at `http://arxiv.org/abs/1103.1819`.

[CNV] J. Cheeger, A. Naber, and D. Valtorta, Critical sets of elliptic equations, *accepted on Comm. Pure Appl. Math.* Available at `http://arxiv.org/abs/1207.4236`.

[HHL98] Q. Han, R. Hardt, and F.-H. Lin, Geometric measure of singular sets of elliptic equations, *Comm. Pure Appl. Math.* **51** (1998), 1425–1443. MR 1639155. Zbl 0940.35065. `http://dx.doi.org/10.1002/(SICI)1097-0312(199811/12)51:11/12<1425::AID-CPA8>3.0.CO;2-3`.

[HL] Q. Han and F.-H. Lin, Nodal sets of solutions of elliptic differential equations. Available at `http://nd.edu/~qhan/nodal.pdf`.

[HL00] Q. Han and F. Lin, Rank zero and rank one sets of harmonic maps, *Methods Appl. Anal.* **7** (2000), 417–442, Cathleen Morawetz: a great mathematician. MR 1869293. Zbl 1006.58011.

E-mail address: `daniele.valtorta@epfl.ch`