Geodesic and translation ball packings in Thurston geometries

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Finding the densest (not necessarily periodic) packing of spheres in the 3-dimensional Euclidean space is known as the Kepler Problem: *No packing of spheres of the same radius has a density greater than the face-centered (hexagonal) cubic packing.* This conjecture was first published by Johannes Kepler in his monograph *The Six-Cornered Snowflake (1611)*, this treatise inspired by his correspondence with Thomas Harriot (see Cannonball Problem). In 1953, László Fejes Tóth reduced the Kepler conjecture to an enormous calculation that involved specific cases, and later suggested that computers might be helpful for solving the problem and in this way the above four hundred year mathematical problem has finally been solved by Thomas Hales. He had proved that the guess Kepler made back in 1611 was correct.

In mathematics sphere packing problems concern the arrangements of non-overlapping identical spheres which fill a space. Usually the space involved is three-dimensional Euclidean space. However, ball (sphere) packing problems can be generalized to the other 3-dimensional Thurston geometries.

In an n-dimensional space of constant curvature $\mathbf{E}^n$, $\mathbf{H}^n$, $\mathbf{S}^n$ $(n \geq 2)$ let $d_n(r)$ be the density of $n+1$ spheres of radius r mutually touching one another with respect to the simplex spanned by the centres of the spheres. L. Fejes Tóth and H. S. M. Coxeter conjectured that in an n-dimensional space of constant curvature the density of packing spheres of radius r can not exceed $d_n(r)$. This conjecture has been proved by C. Roger in the Euclidean space. The 2-dimensional case has been solved by L. Fejes Tóth. In an 3-dimensional space of constant curvature has been investigated by Böröczky and Florian in [2] and it has been studied by K. Böröczky in [1] to n-dimensional space of constant curvature $(n \geq 4)$.

The goal of this talk to generalize the above problem of finding the densest geodesic and translation ball (or sphere) packing to the other 3-dimensional homogeneous geometries (Thurston geometries)

$$\widetilde{\mathbf{SL_2R}},\ \mathbf{Nil},\ \mathbf{S}^2\!\times\!\mathbf{R},\ \mathbf{H}^2\!\times\!\mathbf{R},\ \mathbf{Sol},$$

(see [7], [8], [9], [13], [14], [12]) and to describe a candidate of the densest geodesic ball arrangement. The greatest density up till now is ≈ 0.85327613 whose horoball arrangement is realized in the hyperbolic space $\mathbf{H}^3$. In this talk we show a geodesic ball arrangement in the $\mathbf{S}^2\!\times\!\mathbf{R}$ geometry which density is ≈ 0.87499429 (see [10]).

In [3], [5], [6] and [11] we have studied some new aspects of the horoball and hyperball packings in $\mathbf{H}^n$ and relating to these we will show that the ball, horoball and hyperball packing problem is not settled in the n-dimensional $(n \geq 3)$ hyperbolic space.

E. Molnár has shown in [4], that the homogeneous 3-spaces have a unified interpretation in the projective 3-sphere $\mathcal{PS}^3(\mathbf{V}^4, \boldsymbol{V}_4, \mathbf{R})$. In our work we shall use this projective model to each Thurston geometry.

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