Sraffa and Leontief revisited

Mathematical methods and Models of a Circular Economy

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From natural circular processes to economic circular economy

Life on earth depends on circular processes. This statement is valid for plants, animals and the human being. In early times the human being has found food in nature, without influence on the circular processes. With animal husbandry and agriculture, he has learnt to adapt the life cycles of animals and plants to his own needs, but this type of economy was still cyclical, and the human being has not intervened yet into the circularity of water, of nitrogen and carbon. This has changed with the area of industrial production and exploitation of not renewable natural resources. The many disadvantages of this mode of production, from which the climate change most worries, are now clearly visible. Actually, a new consciousness awakes the interest for the concept of circular economy, which involves the fact that the ‘resources are available in a certain quantity and have an inherent capacity of renewal’, see Aurez [1] (2019), p. 24. In the last century there were two economists who understood and described the economy as a circular process, Wassily Leontief (1906-1999) und Piero Sraffa (1898-1983).

Wassily Leontief (1906-1999) was a Russian American economist, who in 1928 has devoted his doctoral theses to the circularity of economy. He conceived the Input-Output tables (IOT), which today are used in the Statistical Offices of all the countries, in order to establish the annual national production. These tables also contain the results of national accounting.

With his book „*Production of Commodities by Means of Commodities*“ [8], published in the year 1960, Piero Sraffa (1898-1983) explained the basics to a modern understanding of the economic processes of circularity. He considered commodities as means of production and end products. He supposed constant rate of profits and constant wage rates and computed production prices, in a system producing surplus, separated in profits for entrepreneurs and wages for workers.

From agriculture of the ancient civilizations to the classics and neoclassic economies

The existing testimonies of the ancient civilizations of the Egyptians and Mesopotamians in the region of the Fertile Crescent illustrate intensively, that the human beings from the very beginning of history observed the processes of nature. They grasped the cycles of the seasons, the water flow of the Nile, the Euphrates and the Tigris, the influence of weather, and finally the circularity of the life of the plants and the animals. Setting up their agriculture, these early civilizations have observed the circularity of life processes and imitated it. In this way the first economic circular processes have been created. The construction of basic tools followed the same logic. The organisational and artisanal set up led to the flourishment of these early civilizations. Especially the Egyptians have presented on graphic wall pictures the succession in time of the different steps of work in the cultivation of cereals, ploughing, sowing, harvesting, storing, conserving or consuming. The first tools have also been visually presented.

The knowledge of economic circular processes is for these reasons since the ancient civilizations part of our history. For millenaries it has been transmitted from generation to generation and has always been testified by great thinkers and scientists. Thus, the physician and physiocrat François Quesnay (1694-1774) has graphically presented the concept of circularity of economic production processes in his books (1759) as *tableaux économiques* and illustrated with calculations.

There are two events which can be identified, leading to a deviation from the understanding of the circularity of economic processes. As *first event* there is the statement of: Adam Smith (1723-1790) who "asserts the apparently self-contradictory notion that capitalism transforms selfishness into its opposite: regard and service for others", see Foley ([2], p. 2). This mechanism was conducted by the *invisible hand*. When every individual selfishly maximizes his personal utility, then following Smith the wealth of nations is also guaranteed. Thus, "neither Smith nor any of his successors has been able to demonstrate rigorously and robustly how private selfishness turns into public altruism", see Foley [2], p.3).

The concentration on the goal of personal utility leads away from holistic comprehension of economic processes, because the attention is focussed on the end product, this means on the object which gives the psychologically and individually justified utility. The unalterably necessary means of production, as wheat, iron, wood, from which tools are constructed in the economy, as they are presented on the wall pictures of the ancient Egyptian civilization, are put on the background.

The area of *classical political economy* (classics), whose representatives are Adam Smith, David Ricardo and others, based on the comprehension of circularity of economic processes, lasted until 1870. One of its pillars was the *theory of labour value*, which affirms that the price of a commodity depends essentially on the working time necessary for its elaboration. After 1870, and this is the *second event*, the theory of *marginal utility*, which at the beginning only was a theory of the behaviour of consumers, puts the *theory of labour value* in the background. Then, at the eve of the next century some economists concentrated their efforts on the analytical apparatus of the theory of *marginal utility*. They put besides the *marginal utility of the commodities* the *marginal productivity* of the „ factors of production ", which are *labour* and *capital*. This way of arguing is since 1900 the dominating school of thought and is called *neoclassical economy*, even if it has nearly nothing to do with *classics*.

Generally, one considers the decisive difference between *classics* and *neoclassical economy* in its respective price theories, which is also the theory of labour value versus the theory of marginal utility. For our purposes another difference is more important. In the *neoclassical economy* the production process is a one sense road, leading from the production factor labour (L) and capital (K) to an end product (Y) which is generally not precisely defined, thus the process is shortly written as "exploit –

produce – consume – draw away", see also Aurez [1], (2019), p. 24 . This relation is described by the equation *Y = f (L, K)*, where *f* is the „ production function ". What undesired waste goods are generated, from where the labour force comes, and if and how many times the process can be repeated, nobody asks for that, see Knolle [4]. Aurez [1], (2016), p. XIV, speaks of «linear industrial economy».

Sraffa's formal representation of economic production as circular production processes

A detailed word-based presentation of circular production processes, starting from any number of industries and all corresponding parts of means of production tends to the limit of linguistic possibilities. Some authors trying it have been confronted to this difficulty. These linguistic limits are here illustrated with the most elementary wheat example of Sraffa ([8], paragraph 1, p. 21), just comprising two industries and two products.

The next explanations lean on those of Helmut Knolle [4]. Sraffa starts his oeuvre *Production of commodities by means of commodities* [8] with the consideration of a very elementary national economy, consisting of two industries, producing only wheat and iron. With iron we understand simple iron tools, which are unusable after one year and must be replaced. The productivity is so small, that no surplus is generated. There is also no profit. The workers are not paid with money but get wheat as subsistence wages. In the cadre of the chosen technology the specific quantities of wheat and iron in both sectors have to be in a fixed proportion: every miner needs a pickaxe and each agricultural worker needs a scythe, etc. The annual activity is described by following scheme: The first industry produces from 8 tons (t) of iron and 120 quarters (qr) of wheat the quantity of 20 t of iron and, secondly, agriculture produces from 12 t of iron and 280 qr of wheat 400 qr of wheat.

The system uses also in both industries 8+12=20 t of iron and produces the same amount of iron, respectively, uses 120+280=400 qr of wheat and produces the same amount of wheat. The iron industry produces over the own use a surplus of 12 t of iron, which are exchanged against 120 qr of wheat in order to pay subsistence wages to the workers and entrepreneurs. In agriculture one needs just 12 t of iron and can exchange its surplus of 120 qr of wheat with the iron industry.

With this scheme Sraffa has developed a price model, in order to get the equilibrium between the total quantity of means of production and the total final use, occurring over the determination of production prices (exchange values). One observes that 1 t of iron has just the value of 10 qr of wheat, in order that the whole system can be reproduced year after year. The currency of this economy is *wheat*. The workers and entrepreneurs get subsistence wages in wheat.

For agriculture the value of the means of production is equal to the value of the total wheat production: 12 x 10 + 280 x 1=400 x 1 = 400 qr of wheat, for the *iron* industry the value of the means of production is equal to the value of the produced *iron*: 8 x 10 + 120 x 1 = 20 x 10 = 200 qr of *wheat. The total production has a value of 600 qr of wheat*. One can formulate this equivalence in another way: Agriculture has to deliver at the end of the year the quantity of 120 qr of wheat of own production against 12 t of iron from the *iron* industry, which needs itself 120 qr of *wheat* and can thus deliver 12 t of *iron* to agriculture.

Agriculture needs the remaining 280 qr of wheat partly as seeds, partly to pay its workers and entrepreneurs, and the 8 t of iron as iron tools for the next annual production. The iron industry needs the obtained 120 qr of wheat, in order to pay subsistence wages for its workers and entrepreneurs. The remaining 8 t of iron are used as iron tools for the own production process. The system is in equilibrium, when the inputs and outputs are equal, thus, 10 qr of wheat have to be exchanged against 1 t of iron.

This was a verbal description of the most elementary example of Sraffa. Here, there is no production of surplus, no profit for entrepreneurs, no wages for workers, there are exclusively subsistence wages for all the active persons. The complexity of a verbal description of a circularity process has become visible throughout is example. This seems indeed to be a reason why the detailed concept of circularity is apparently uncommon to a larger public. For economists who exclusively use word-based descriptions and not number-based matrix algebra, the accessibility is seemingly also restricted or difficult.

With the means of matrix algebra, one succeeds on the other hand to describe the circular processes in a very elegant way.

In the next section we show the contribution of the book **Sraffa and Leontief revisited**.

From Sraffa's Elementary Example to the SWISS-IOT 2014 (49 sectors/product groups)

Rigorous matrix algebra representation of Sraffa's circular economy

Sraffa's oeuvre «*Production of commodities by means of commodities*» [8] (1960) has met great interest soon after its publication among famous economists (Pasinetti, Schefold) and mathematicians (Newman). Since, a great deal of work has been accomplished to complete Sraffa's book. Thus, Peter Newman and Ann Arbor [6] have explicitly shown in their article, that the theorem of Perron-Frobenius is the algebraic basis of the Sraffa price model with surplus, which is transformed in profits and where the workers obtain subsistence wags. Indeed, Newman has correctly shown here that the calculation of throughout positive prices leads to an eigenvalue problem. Further, both authors have also solved the general Sraffa price model with wages for workers and formulated the conditions for the existence of the appearing inverse matrices, referring to theorems in Felix R. Gantmacher [3] and others, in order to obtain also here throughout correctly positive prices as solutions.

In the year 1976 Bertram Schefold has once more reformulated and mathematically completed Sraffa's basic theses in an appendix of the German edition of Sraffa's book [8].

Our intention is to complete the existing literature (Pasinetti, Schefold, Kurz *and* Selvadori) by number-based and application-oriented presentations, using throughout modern matrix notations, referring to the specific theorems of matrix algebra. For the calculation of numerical examples and applications on the Input-Output Tables of Germany, Switzerland and other countries, the software packages MATHEMATICA and MATLAB have been used, relying on the calculus and graphic facilities of these tools. In this sense, there is a line starting from Sraffa's circularity models to their formal description by matrices, referring to the needed Theorems, to the calculations with the help of modern software and the interpretation of the numerical results.

Our experience shows that without a rigorous application of matrix algebra it is impossible to describe completely the circular economy as a closed system in any dimension. The above verbal description of the most elementary Sraffa price model illustrates precisely this fact. It is essential to explain what domains of mathematics are on the basis of the calculations that are realized here: It is the *group of the Perron-Frobenius theorems*. In our book we completed the variants of the different Lemmas and Theorems belonging to this group of propositions, and we associated the different proves, when this seemed to be necessary. We have been inspired by the standard oeuvre of Gantmacher [3].

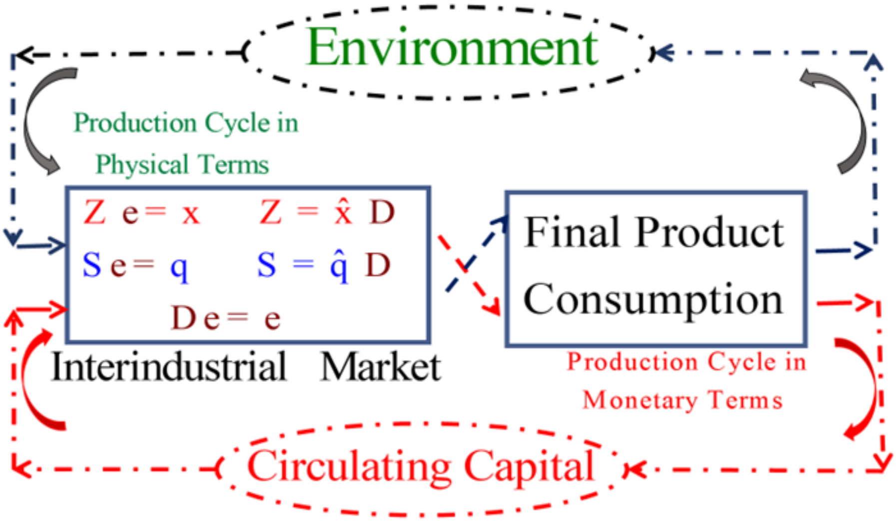
The following model concept has been introduced to underline the applicability on free markets. Every commodity, for example wheat or iron, are characterized in a free economy by **four attributes**: the quantity, the price, the value and the object. The term object has to be understood in such a way that there exists in a free economy for each commodity more than one realisation, so that the buyer is free to choose one of the possible *item / industry / branch*. This typically happens with the acquisition of a bicycle. Finally, the buyer decides to acquire a precise type of bicycle, an individually realized item or object from a specific trade mark. For each purchase and sale of a commodity there are therefore four degrees of freedom: the choice of the object e, the price p, the quantity q and the value x. This fundamental concept leads to a renewal of the algebraic presentation.

11'237	1	1-3	Agriculture, forestry and fishing
7'149	2	3-5	Mining and quarrying
18'191	3	9-10	Manufacture of food and tobacco products
3'965	4	13 - 15	Manufacture of textiles and apparel
9'604	5	16	Manufacture of wood and of products of wood and cork, except furniture
4'091	6	17	Manufacture of paper and paper products
3'599	7	18	Printing and reproduction of recorded media
23'046	8	19 - 20	Manufacture of coke, chemicals and chemical products
44'786	9	21	Manufacture of basic pharmaceutical products and Pharmaceutical preparations
7'494	10	22	Manufacture of rubber and plastic products
8'322	11	23	Manufacture of other non-metallic mineral products
11'527	12	24	Manufacture of basic metals
18'047	13	25	Manufacture of fabricated metal products, except machinery
			And equipment
33'175	14	26	Manufacture of computer, electronic and optical products
12'286	15	27	Manufacture of electrical equipment
7'728	16	28	Manufacture of machinery and equipment n.e.c.
1'895	17	29	Manufacture of motor vehicles, trailers and semi-trailers
2'030	18	30	Manufacture of other transport equipment
650	19	31	Manufacture of furniture
2'042	20	32	Other manufacturing
4'144	21	33	Repair and installation of machinery and equipment
32'470	22	35	Electricity, gas, steam and air-conditioning supply
4'891	23	36 - 39	Water supply, waste management
23'831	24	41 - 43	Construction
5'829	25	45	Wholesale and retail trade and repair of motor vehicles and motorcycles
64'563	26	46	Wholesale trade, except of motor vehicles and motorcycles
3'829	27	47	Retail trade, except of motor vehicles and motorcycles
38'045	28	49 - 51	Land, water and air transport and transport via pipelines
21'767	29	52	Warehousing and support activities for transportation
4'687	30	53	Postal and Courier Aktivitas
1'318	31	55	Accommodation
1'996	32	56	Food and beverage service activities
7'040	33	58 - 60	Publishing, audiovisual and broadcasting activities
8'845	34	61	Telecommunications
13'488	35	62 - 63	IT and other information services
34'117	36	64	Financial service activities
8'652	37	65	Insurance
10'162	38	68	Real estate activities
64'390	39	69 - 71	Legal, accounting, management, architecture, engineering
			activities
1'567	40	72	Scientific Research and Developern
7'232	41	73 - 75	Other professional, scientific and technical activities
34'633	42	77 - 82	Administrative and support service activities
4'988	43	84	Public administration
6'945	44	85	Education
1'138	45	86	Human health activities
2'664	46	87 - 88	Residential care and social work activities
4'836	47	90 - 93	Arts, entertainment and recreation
3'511	48	94 - 96	Other service activities
	49	97 - 98	Activities of households as employers of domestic
0			personnel / Undifferentiated goods- and services-producing
			activities of private households for own use
652`440			TOTAL

Designations of the sectors of the SWISS IOT 2014

This concept of four attributes has been applied to symmetric national Input-Output Tables (IOT), which are composed of sectors (homogeneous CPA branches), producing CPA product groups. Sraffa's industries correspond to these sectors, Sraffa's products to the CPA-product groups. The abbreviation CPA means: Statistical Classification of Products by Activity in the European Economic Community.

Actually, one had in general only two levels of matrices, one for the commodity flow matrix and one for the coefficient's matrix. This structure has to be enlarged to four levels (4 Input Output matrices **Z**, **T**, **S**, **D**, 4 state matrices **A**, **B**, **C**, **D**, 4 market activity vectors **x**, **p**, **q**, **e**). This mathematical completion and rounding off are developed in our book and presented in graphical form as follows:



Sraffa`s production of commodities by means of commodities (PCMC) with the production cycles in physical terms and in monetary terms. The row-sums of the matrices **Z** and **S,** and the stochastic production matrix **D** of the inter industrial production determine **q** and **x**. The components of the vector **x** represent the values of inter industrial production, while the components of the vector **q** represent the quantity of each product group. The elements of the technology matrix **D** represent the influence of every sector on the technology of the other sectors.

The Stochastic Similarity Table of Interindustrial Production

		Row-sum	PF-Eigenvector	Stochastic similarity	I/O matrices
Value	x	$Ze = x$	$Ax = x$	$A = \hat{x}\, D\, \hat{x}^{-1}$	$Z = \hat{x}\, D$
quantity	q	$Se = q$	$Cq = q$	$B = \hat{p}\, D\, \hat{p}^{-1}$	$T = \hat{p}\, D$
price	p	$Te = p$	$Bp = p$	$C = \hat{q}\, D\, \hat{q}^{-1}$	$S = \hat{q}\, D$
Object	e	$De = e$	$De = e$	$D = \hat{e}\, D\, \hat{e}^{-1}$	$D = \hat{e}\, D$

The matrix and vector relationships of the inter industrial market. The elements of the Input-Output matrix **Z** indicate how the value vector **x** is composed, while the elements of the matrix **T** determine how the price vector **p** is composed. The same statement is valid for the elements of the matrix **S** determining the dependence of the production quantity vector **q** on the production of other sectors. The positive vectors **x**, **p**, **q** and the technological object vector **e** are the **Perron-Frobenius eigenvectors** (**PF-eigenvector in the Table**) of the corresponding state matrices **A**, **B**, **C**, and **D**.

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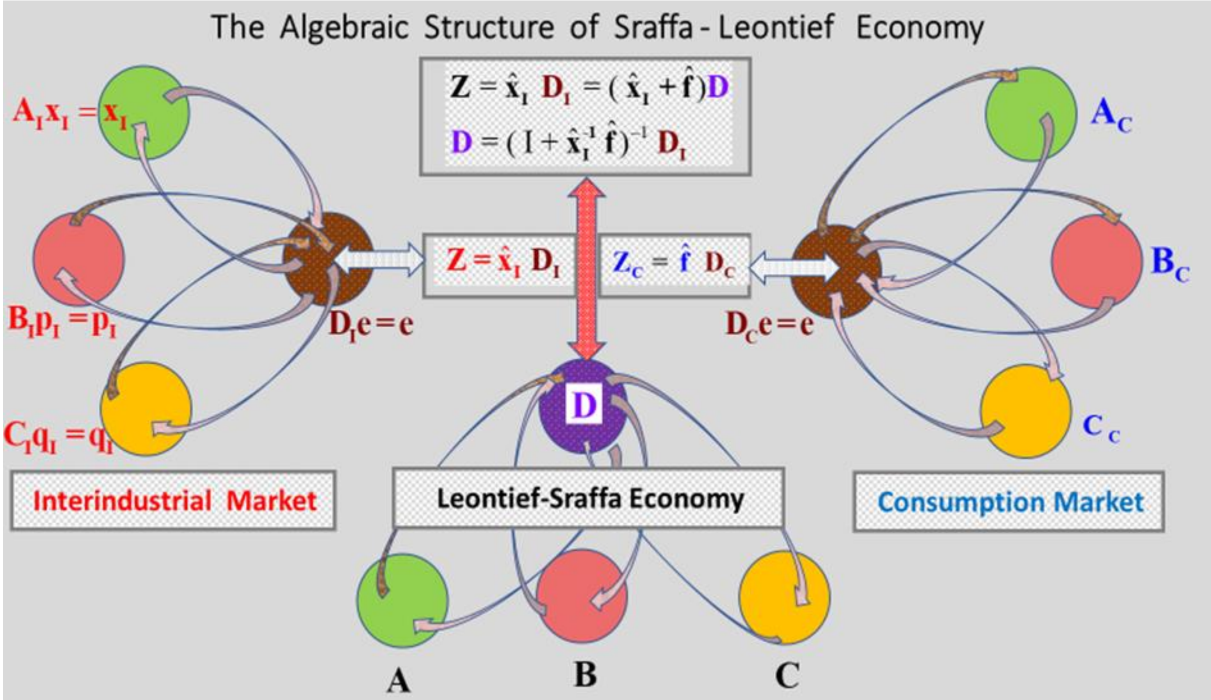
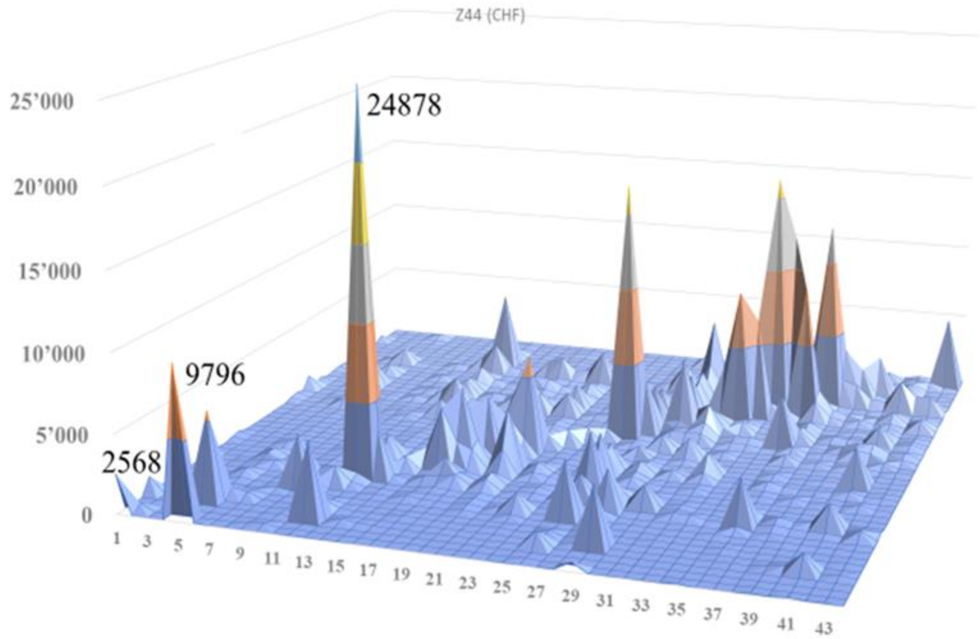
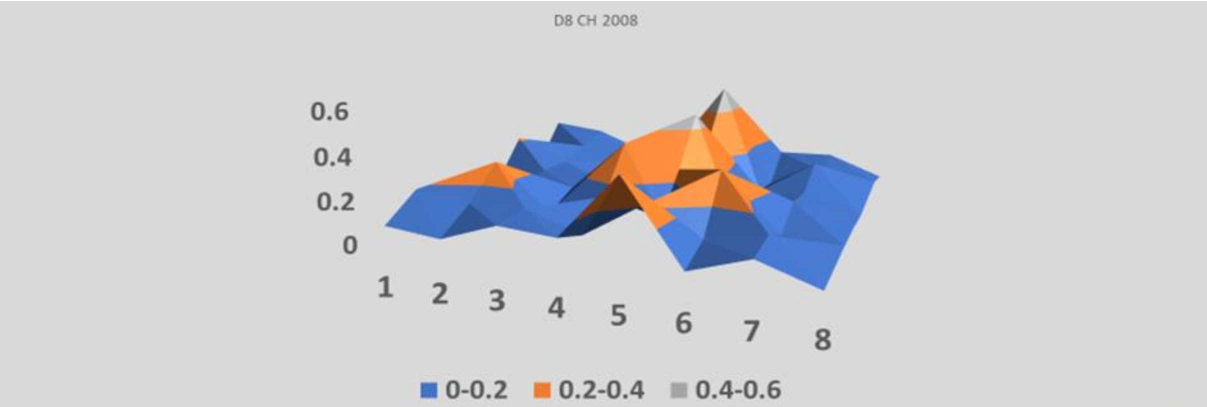


Graphical illustration of the algebraic modelling of the Sraffa-Leontief economy and its relationships with the inter industrial and the consumption markets. The state matrices **A**, **B**, **C** and **D** of the markets are illustrated together with the corresponding Perron-Frobenius eigenvectors of value **x**, price **p**, **quantity q and item technology vector e.**



Graphical illustration of the Input-Output Matrix **Z** (44 sectors / each sector produces one CPA product group of exactly one price) of the Swiss Inter industrial production 2008 (510.79 Billion CHF).



0.08579054	0.05404462	0.14273563	0.11888518	0.41310235	0.03780734	0.12119121	0.02644311
0.1853483	0.14401581	0.14715457	0.04949163	0.21003866	0.09016263	0.14065689	0.0331315
0.00213577	0.27759712	0.05576718	0.1548689	0.0814699	0.33061815	0.05878867	0.03875431
0.02570365	0.07891532	0.10866624	0.35184796	0.24685924	0.0099385	0.13886491	0.03920419
0.02922004	0.11135184	0.09226702	0.0993736	0.46220687	0.06595459	0.1137422	0.02588384
0.21143186	0.09999559	0	0.1172798	0.30051268	0.03127852	0.19639294	0.04310861
0.01995973	0.13284852	0.06966223	0.10939912	0.50914992	0.00733287	0.09160123	0.06004637
0.2008815	0.16885424	0.06663077	0.0623989	0.22762451	0.12199166	0.12654652	0.0250719

Value: CH 2008 **x = A x = Z e**

			Z 6				x6
0.4926	0.0792	0.2400	0.2015	0.0198	0.0440		1.0774
0.0334	0.9805	0.0020	0.0370	0.0186	0.0182		1.0900
0.0752	0.0469	0.6401	0.0923	0.0165	0.0337		0.9051
0.0303	0.0723	0.0023	0.1822	0.0007	0.0285		0.3167
0.0155	0.0445	0.0078	0.0246	0.6656	0.0141		0.7723
0.0042	0.0586	0.0002	0.0179	0.0027	0.0844		0.1683

			A 6				x6
0.4572	0.0727	0.2652	0.6364	0.0257	0.2614		1.0774
0.0310	0.8995	0.0022	0.1170	0.0241	0.1083		1.0900
0.0698	0.0431	0.7072	0.2916	0.0214	0.2006		0.9051
0.0282	0.0664	0.0026	0.5753	0.0010	0.1694		0.3167
0.0144	0.0409	0.0086	0.0776	0.8618	0.0838		0.7723
0.0039	0.0538	0.0002	0.0566	0.0035	0.5019		0.1683

Object: D e = e

			D6				e
0.4573	0.0736	0.2228	0.1871	0.0185	0.0408		1
0.0307	0.8996	0.0019	0.034	0.0171	0.0167		1
0.0832	0.0519	0.7072	0.102	0.0183	0.0373		1
0.096	0.2286	0.0075	0.5754	0.0025	0.0901		1
0.0202	0.0577	0.0101	0.0319	0.8618	0.0183		1
0.025	0.3485	0.0016	0.1066	0.0164	0.5019		1

Up: graphical illustration of the Input Output matrices **Z8** and **D8** - in table form - of the 8 largest sectors of the SWISS IOT 2008, normalized by the Perron-Frobenius (PF) eigenvalue of the official Input Output table SWISS-IOT-2008. Down: The matrices **Z6, A6**, **D6** of the six largest sectors. The value vector **x6** is the Perron-Frobenius eigenvector of the state matrix **A6**. It is simultaneously the row-sum of the matrix **Z6**. The matrix **D6** is a right stochastic matrix. Its eigenvector is the object vector **e**. The elements of the technology matrix **D6** indicate how the technology of one sector is participating to the production technology of the other 5 sectors.

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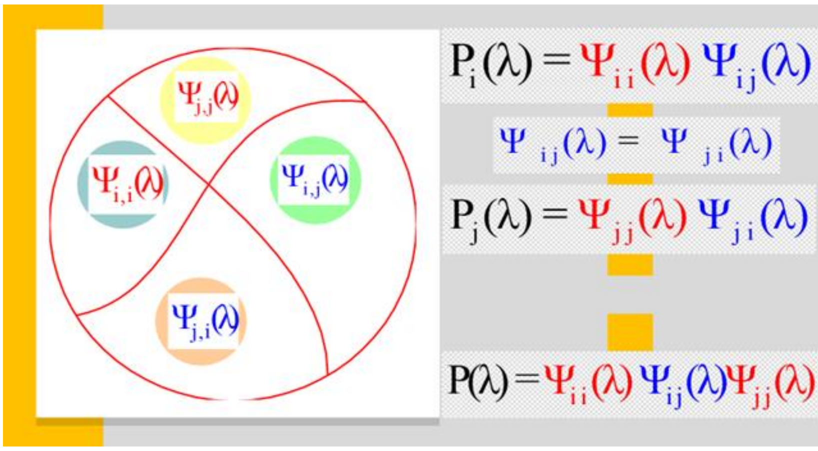
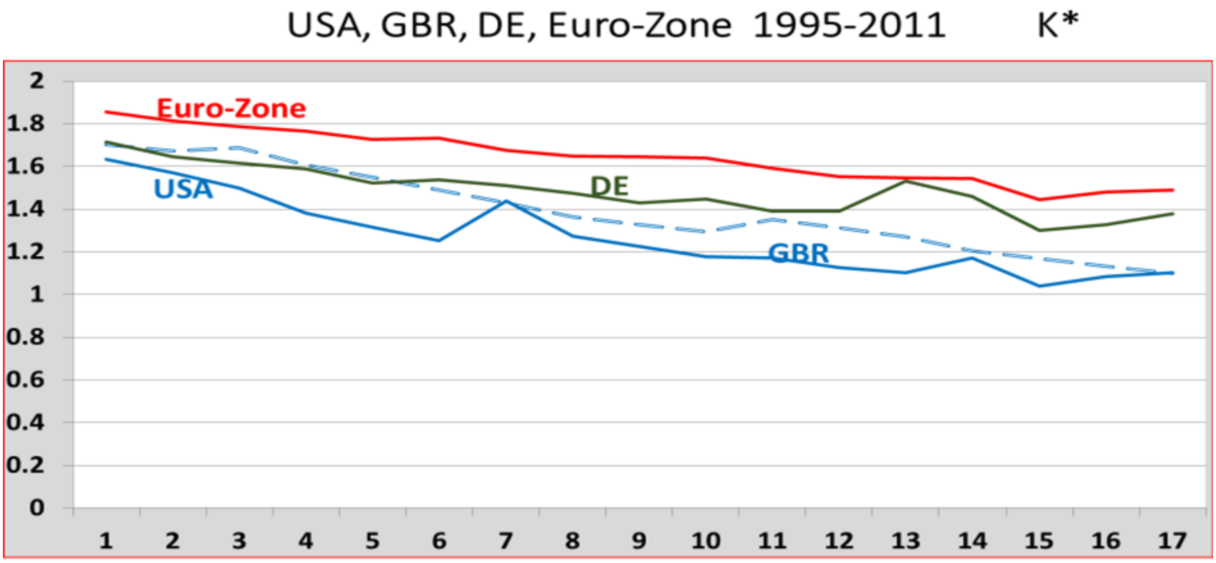
Sraffa`s basic industries, respectively basic products

Sraffa [8] distinguished in circular economies of *n* industries and *n* products, the case of *single product industries*, where each *industry* produces only one *product,* and the case of *multiple-product industries* (joint production), where each industry produces one or more *product*. In both cases, there is the fundamental economic notion of *basic industries*, respectively *basic products,* which are those indispensables for a given economy, as the branch of agriculture, producing, wheat, wood, vegetables, and guarantee the existence of the whole system. On the opposite, there are the *non-basic industries*, which may – historically - produce not indispensable luxuries, as jewellery and race horses. An important question is the determination of the number *m* of *non-basic industries* among the total number *n* of industries, leading to the number *n-m* of basic industries. Sraffa ([8], Par. 6, 60) has given in the case of *single product industries* and *joint production*, specific methods to determine the number *m*. Schefold [7] has reformulated Sraffa's method for joint production in terms of matrix algebra.

In our book we have also picked up Schefold's matrix method and formulated a *matrix rank criterion* which allows to determine the number *m*. We have also put together other methods to determine the number *m*, as the calculation of the matrix proposed by Pasinetti, and illustrated the methods with many numerical examples. We have also identified the origin of this matrix methodology as an idea, going back to Lev Semyonovich Pontryagin (1908-1988) and is applied extensively in the system control theory as the controllability-observability conditions. This is a highly mathematical concept which we present in the figure below with subsequent explanations.



Graphical illustration of *basic industries* (The Sraffa notion *industry* can be replaced by *sect*or, if the application concerns IOTs): The industry *i* generates through its corresponding vectors $\mathbf{x}_i$, $\mathbf{p}_i$, $\mathbf{q}_i$, $\mathbf{e}_i$ a cyclic space of influence represented through its characteristic polynomial $P_i(\lambda)$. The same is valid for industry *j* with the characteristic polynomial $P_j(\lambda)$. If the two industries influence other industries, then, the dimension of all influenced industries- including their own cyclic spaces - is given by the product $P(\lambda)$ of three polynomials. The two polynomials, associated to industries, are therefore not relatively prime. The **blue polynomial $\Psi_{I,j}(\lambda)$ or $\Psi_{j,i}(\lambda)$ represents** the space influenced by both industries. If the dimension of $P(\lambda)$ is *n*, then, the two industries are *basic*. If the dimension of $P_i(\lambda)$ is *n*, then the corresponding state matrix has a simple (cyclic) structure and can be transformed to a Frobenius form (phase state form). In such a case the polynomial coefficients are directly the coefficients of the Frobenius form. The origin of these conditions goes back to L. S. Pontryagin. There are already efficient algorithms that determine not only the influence space, but also determine the coefficients of influence over the state matrices **A**, **B**, **C** and **D**. Further any appropriate convex combination of suggested influence vectors can be tested about their **influence**.



The monthly operating/circulating capital K* (normalized according to the corresponding Perron-Frobenius-(PF) eigenvalue of the flow commodity matrix of the Input-Output table of the country) for the USA, GBR, DE, Euro-Zone over 17 years (1995-2011).

With this book Sraffa and Leontief revisited we want to close the gap in the literature just described above.

The book contains many different interesting results, some of which will be summarized here.

a) Hassan. A. Nour Eldin achieved a complete analysis of the algorithmic properties of the inter industrial market. The inter industrial economy is formulated as a boundary value problem, where the vectors of value x, price p, quantity q and object e are correspondingly the Perron-Frobenius eigenvectors of the state matrices **A**, **B**, **C** and **D**. They are simultaneously the row-sum of the commodity flow matrix **Z**, the matrices **T, S** and **D**. The technology matrix **D** is a right stochastic matrix. It is the stochastic similarity matrix of the state matrices **A**, **B** and **C**. Its elements influence therefore all other matrices and vectors.

b) There is the fundamental economic notion of *basic industries*, respectively *basic products*, which describe those industries indispensables for an economy, as the branch of agriculture, producing, wheat, wood, vegetables, and guarantee that the whole system holds. On the opposite, there are the *non-basic industries*, respectively *non-basic products*, which may produce not indispensable luxuries, as jewellery and race horses. Our book presents a comprehensive *matrix rank criterion* to determine the number of *non-basics*. Thus, one can easily determine the number of *basics* in a Sraffa price model.

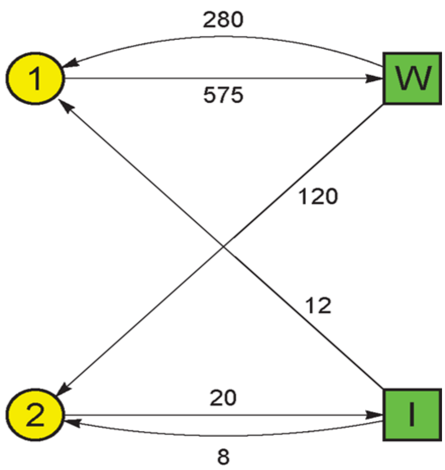
c) The Sraffa price model, where the surplus is not divided into profit and wages, offers the possibility to compute a measure of productivity *R* of the described economy. The commodity flow matrix of a symmetric national Input-Output Table (IOT) has been taken as basis of such a Sraffa price model. Hassan A. Nour Eldin could show that the mentioned productivity measure *R* can be computed, without need to determine the prices of the Sraffa model. It also shows that we are here in presence of a boundary value problem, as they are known from physics. Thus, one has shown that the *Sraffa price model* can be combined with *Input-Output Tables* to a dynamic system.

d) Helmut Knolle extended the Sraffa price model in such a way that recycling of waste products can be involved and nevertheless the economy turns out generating profits.

Sraffa and Leontief revisited, mathematical Methods and Models of a Circular Economy, de Gruyter,

e) Jean-François Emmenegger has shown that the Sraffa price model can be extended in a way that every industry has its own rate of profits and its own wage rate, thus, one can work with real distributions of wage rates and profit rates.

f) Daniel Chable has proposed the extensive use of *graphs* and *directed graphs* to represent the production schemes of Sraffa price models, as shown below. They are called *Sraffa networks*. We show such a network for the third elementary example of Sraffa ([8], paragraph 5), developed form the above presented elementary example, but with a total production of 575 qr of wheat. Circles represent the industries and squares the products. We have also adapted graph-theoretical criteria to determine the presence of *basics* and *non-basics* in graphs and directed graphs, representing production schemes of Sraffa price models.



Our book offers a new perspective for the understanding of the inner nature of circular economy. Matrix algebra reveals to be the ideal tool to describe circular economy of commodities, which appear ones as means of production and ones as final products. There are also numerous didactical examples and applications to the Input-Output Tables (IOT) of different countries set up with complete calculations of the solutions. The Theorems and Lemmas with the associated proves, necessary to understand the different aspects and variants of Sraffa price models, respectively to grasp the mathematical basis, are presented in completed form.

Bibliography

[1] Aurez, Vincent, Georgeault Laurent, *Economie circulaire, Système économique et finitude des ressources*, De Boeck Supérieur s. a., Louvain-la-Neuve, (2016) and (2019)

[2] Foley, D. K., Adam's Fallacy*, A guide to economic Theology*, The Belknap Press of Harvard University Press, 2006.

[3] Gantmacher, F., R., *Matrizentheorie*, Springer-Verlag, Berlin, (1986).

[4] Knolle, Helmut, *Die Wachstumsgesellschaft, Aufstieg, Niedergang und Veränderung*, Köln, PapyRossa, (2016).

[5] Leontief, Wassily Wassiljewitsch, *Die Wirtschaft als Kreislauf*, Archiv für Sozialwissenschaft und Sozialpolitik, 60, pp. 577--623, Berlin, (1928).

[6] Newman, Peter, Arbor Ann, *Production of commodities by means of commodities*, Schweizerische Zeitschrift für Volkswirtschaftslehre und Statistik, 98. Jg., pp. 58--75, (1962).

[7] Schefold, Bertram, '*Mr. Sraffa on Joint Production and other Essays*", Routledge, (1989).

[8] Sraffa, Piero, *Production of commodities by means of commodities*, Cambridge University Press, Cambridge (1960), and the German edition, Verlag Suhrkamp 780, (1976), with an appendix of Bertram Schefold.