Lecture course: Metric projective geometry

Concept:

- Explain as much as possible of modern research mathematics on relatively simple examples.
- Prove at least one “easy to formulate, hard to prove without the theory, easy to prove” theorem in each lecture.

Plan:

1.
   - Definition and basic properties of projective structure. Application: Isometries of Hilbert metrics
   - Projective invariant equations. Application: Topology in the 2-dimensional case
2.
   - Metrization equation and local normal forms in dimension 2. Application: Solution of the problems stated by Sophus Lie in 1882.
3.
   - Projectively invariant tensors. Application: Proof of projective Lichnerowicz conjecture
   - If time allows: metrics with higher degree of mobility and solution of Weyl-Ehlers problem

What is a projective structure?

Informal and inefficient definition: *Projective structure* is a $\underbrace{\text{sufficiently big}}_{\text{to be explained}}$ family of curves that after a reparameterization are geodesics of some affine connection.

Sufficiently big: In any point for any direction there exists a precisely one curve from the family passing through this point and tangent to this direction.

Simplest example: the set of all straight lines on $\mathbb{R}^2$: it is sufficiently big, and they are geodesics of the flat (and not only of the flat) connection.

General example: Take any affine connection $\nabla = (\Gamma^i_{jk})$ and consider the family of its (geometrically different) geodesics (parameterized by any parameter)

Efficient definition of projective structure requires a bit of theory

Let us first study the following question: Suppose we have two symmetric affine connections, $\nabla = (\Gamma^i_{jk})$ and $\bar\nabla = (\bar\Gamma^i_{jk})$. When each geodesic of ∇, possibly after a reparameterization, is a geodesic of $\bar\nabla$?

Def. Two connections are ∇ and $\bar\nabla$ said to be projectively equivalent, if any geodesic of ∇, possibly after a reparameterization, is a geodesic of $\bar\nabla$.

Theorem 1 (deep classics: Levi-Civita 1896, Weyl 1924). $\nabla = (\Gamma^i_{jk})$ is projectively equivalent to $\bar\nabla = (\Gamma^i_{jk})$, if and only if there exists an 1-form $\phi = \phi_i$ such that

$$\bar\Gamma^i_{jk} = \Gamma^i_{jk} + \phi_k\delta^i_j + \phi_j\delta^i_k. \qquad (*)$$

The condition $(*)$ **in the index-free form:** for any vector X and any vectorfield V,

$$\bar\nabla_X V - \nabla_X V = \phi(X)V + \phi(V)X \qquad (**)$$

Easy control question to the audience:

Theorem 1 (deep classics: Levi-Civita 1896, Weyl 1924). $\nabla = (\Gamma^i_{jk})$ is projectively equivalent to $\bar\nabla = (\Gamma^i_{jk})$, if and only if there exists an 1-form $\phi = \phi_i$ such that

$$\bar\Gamma^i_{jk} = \Gamma^i_{jk} + \phi_k\delta^i_j + \phi_j\delta^i_k. \qquad (*)$$

Question: Does there exist two projectively equivalent symmetric affine connections on $\mathbb{R}^2$ (or $\mathbb{R}^n$) such that they coincide in some neighborhood and are different in another neighborhood?

Of course YES!!! Take a 1-form which is not zero in a neighborhood but is zero outside the neighborhood, take any connection $\nabla = (\Gamma^i_{jk})$ and "deform" it by $(*)$.

Proof of Theorem 1 in your first exercise for the exercise session, let me give some hints:

Hint 1. Show that Any reparameterised geodesic satisfies the equation $\nabla_{\dot\gamma}\dot\gamma = \alpha(t)\dot\gamma$ for a certain function $\alpha(t)$. Any regular curve satisfying this equation is a reparameterized geodesic.

For physicists the statement in Hint 1 is trivial: $\nabla_{\dot\gamma}\dot\gamma$ is an acceleration, so geodesics are the trajectories of particles with no acceleration (= *freely falling particles*). The condition $\nabla_{\dot\gamma}\dot\gamma = \alpha(t)\dot\gamma$ means that at every point the acceleration is proportional to the velocity, which implies that the particles go along the same trajectory as in no acceleration case but the speed is not constant.

Hint 2. Use the formula

$$\bar\nabla_X V - \nabla_X V = \phi(X)V + \phi(V)X \qquad (**)$$

and Hint 1 to show that the geodesics of ∇ are reparameterized geodesics of $\bar\nabla$.

Hint 3. The statements formulated in Hint 1 and Hint 2 imply Theorem in direction $\Longrightarrow$. In order to show the $\Longleftarrow$-direction, repeat first the proof of the statement from basic linear algebra that quadratic function determines the corresponding symmetric quadratic form.

Definition of projective structure

Informal and inefficient definition: *Projective structure* is a sufficiently big family of curves that after a reparameterisation can be geodesics of some affine connection.

Efficient Def. By *projective structure* we understand the equivalence class of symmetric affine connections with respect to the equivalence relation "projective equivalence": two symmetric affine connection correspond to one projective structure, iff their difference equals $\phi_k\delta^i_j + \phi_j\delta^i_k$ for an 1-form ϕ.

2-dim projective structures and second order ODEs

In dimension $n = 2$, because of the symmetries $\Gamma^i_{jk} = \Gamma^i_{kj}$, the components of Γ^i_{kj} in coordinates are $\frac{n^2(n+1)}{2} = 6$ function. The freedom in choosing the connection in the projective class is $n = 2$ "functions" (ϕ_1, ϕ_2). Thus, locally, projective structure is given by 4 functions of the coordinates. Let us give, following Beltrami 1859, a geometric sense to these 4 functions.

Theorem 2. Let $\left[\Gamma^i_{jk}\right]$ be a projective structure on $U \subset \mathbb{R}^2(x,y)$. Consider the following second order ODE

$$y'' = \underbrace{-\Gamma^2_{11}}_{K_0} + \underbrace{(\Gamma^1_{11} - 2\Gamma^2_{12})}_{K_1} y' + \underbrace{(2\Gamma^1_{12} - \Gamma^2_{22})}_{K_2} y'^2 + \underbrace{\Gamma^1_{22}}_{K_3} y'^3. \qquad (1)$$

Then, for every solution $y(x)$ of (1) the curve $(x, y(x))$ is a (reparametrized) geodesic.

Corollary. The coefficients $K_0, \ldots, K_3$ of ODE (1) contain all the information of the projective structure: two connections are projectively related iff the corresponding functions $K_0, \ldots, K_3$ coincide.

2nd Exercise. Prove the Corollary: show that the kernel of the (linear) mapping $\Gamma^i_{jk} \mapsto (K_0, K_1, K_2, K_3)$ consists of tensors $T^i_{jk} = \phi_k \delta^i_j + \phi_j \delta^i_k$.

Theorem 2. Let $\left[\Gamma^i_{jk}\right]$ be a projective structure on $U \subset \mathbb{R}^2(x,y)$. Consider the following second order ODE

$$y'' = \underbrace{-\Gamma^2_{11}}_{K_0} + \underbrace{(\Gamma^1_{11} - 2\Gamma^2_{12})}_{K_1} y' + \underbrace{(2\Gamma^1_{12} - \Gamma^2_{22})}_{K_2} y'^2 + \underbrace{\Gamma^1_{22}}_{K_3} y'^3. \qquad (1)$$

Then, for every solution $y(x)$ of (1) the curve $(x, y(x))$ is a (reparametrized) geodesic.

Example. The flat projective structure $[\Gamma^i_{jk} \equiv 0]$ corresponds to the ODE $y'' = 0$. The solutions of this ODE are $y(x) = ax + b$, and the curves $x \mapsto (x, y(x)) = (x, ax + b)$ are indeed straight lines.

Remark 1. Note that the set of curves of the form $(x, y(x))$ is quite big: at any point in any direction there exists such a curve passing through this point in this direction.

Remark 2. We see a special feature of geodesics of affine connections: they are essentially the same as solutions of 2nd order ODE $y'' = F(x, y, y')$ SUCH THAT THE RIGHT HAND SIDE IS POLYNOMIAL IN y' OF DEGREE ≤ 3.

3rd exercise: Find a family of curves such that it is sufficiently big in the sense above but is not a projective structure. Do not forget about curves tangent to vertical direction and about that the family shold be smooth!

How many geodesics determine the projective structure?

We will answer this question in DIMENSION 2, and give an application in the next slides.
We consider a projective structure $\left[\Gamma^i_{jk}\right]$ and the corresponding ODE

$$y'' = \underbrace{-\Gamma^2_{11}}_{K_0} + \underbrace{(\Gamma^1_{11} - 2\Gamma^2_{12})}_{K_1} y' + \underbrace{(2\Gamma^1_{12} - \Gamma^2_{22})}_{K_2} y'^2 + \underbrace{\Gamma^1_{22}}_{K_3} y'^3. \quad (1)$$

Claim. For any point $(\hat{x}, \hat{y})$, 4 different geodesics passing through this point determine the coefficients $K_0(\hat{x}, \hat{y}), K_1(\hat{x}, \hat{y}), K_2(\hat{x}, \hat{y}), K_3(\hat{x}, \hat{y})$ at this point.

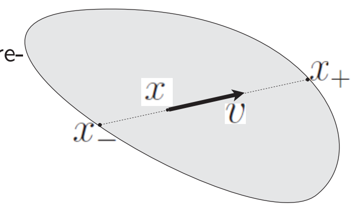
Proof. 4 different geodesics passing through $(\hat{x}, \hat{y})$ correspond to 4 different solutions y_1, y_2, y_3, y_4 of (1) such that $y_i(\hat{x}) = \hat{y}$. Knowing geodesics implies that we know $y_i'(\hat{x})$ and $y_i''(\hat{x})$ which implies that we known the values of the polynomial

$$P(y') = K_0(\hat{x}, \hat{y}) + K_1(\hat{x}, \hat{y})y' + K_2(\hat{x}, \hat{y})y'^2 + K_3(\hat{x}, \hat{y})y'^3$$

at four points $y_1'(\hat{x}), ..., y_4'(\hat{x})$ and we know that values at 4 points determines a polynomial of degree ≤ 3. □

Application in Finsler geometry: proof of the 2-dim de la Harpe conjecture

Let $K \subset \mathbb{R}^n$ be a compact convex body.



Hilbert metric is a Finsler metric, the corresponding Finsler function is given by

$$F(x, v) = \frac{|v|}{|x - x_+|} + \frac{|v|}{|x - x_-|}.$$

- Straight line segments are geodesics
- Projective transformations preserving the convex body preserve the Hilbert metric

Remark. If the boundary is not strictly convex, the geodesics are not necessary unique.

Question of de la Harpe (1991)

For what K all isometries of K are projective transformations?

- Example (de la Harpe): If K is simplex, there exist isometries that do not come from projective transformations.
- If K is strictly convex, any isometry is a projective transformation (deep classics; possibly Hilbert)
- 2011: Answer by Walsh and Lemmens for polyhedral K.
- 2013: Answer by Walsh for all convex bodies

Theorem (M∼ – Troyanov 2016(2013)). In dimension two, each isometry $\phi : K \to K$ is a projective transformation unless K is a triangle.

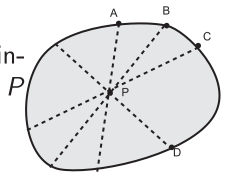
We see that our theorem is not new; but the proof of Walsh is relatively complicated and you will see that our proof is trivial for those who heard the first part on today's lecture.

Proof .

Fact (possibly Hilbert 1895, de la Harpe 1991). A straight line containing an extremal point is UNIQUE MINIMIZING geodesic.

We consider four extremal points A, B, C, D of K.

For every point $P \in int(K)$, we consider the intersections of the straight lines containing A and P (resp., B and P, C and P, D and P).



As we learned today, these four straight lines define uniquely a projective structure; this projective structure is projectively flat (all geodesics are straight lines)

The push-forward of this projective structure is a projective structure, since the 4 geodesics are unique, isometry map sends them to straight lines and therefore the push-forward of the projective structure is projectively flat and ϕ is a projective transformation □

Projectively invariant differential operators

PLAN

1. Definition
2. Two main examples: (projective) Killing and metrization equations
3. Philosophy of metric projective geometry
4. Application: what 2 dim manifolds admit projectively equivalent metrics.

Projectively invariant operators: def and trivial examples

Projectively invariant = does not depend on the choice of a connection in the projective class and on the coordinate system.

Not an Example. Covariant differentiation of vectors or tensors is NOT projectively invariant: If we replace ∇ by a projectively equivalent $\bar\nabla$, then the covariant derivative will be CHANGED:

$$\bar\nabla_X V - \nabla_X V = \phi(X)V + \phi(V)X.$$

Trivial Example. The outer derivative $\omega \mapsto d\omega$ on the space of k-forms is projectively invariant. Indeed, it does not depend on a connection at all. (Say, for 1-forms, $d(adx + bdy) = \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right) dx \wedge dy$)

Our next goal is to construct two 'nontrivial' projectively invariant differential operations; they will play an important role later today and in other lectures but the price we need to pay now is that we need to introduces weighted tensor fields

What are weighted tensors? What is weight?

We assume that our manifold M is orientable and fix an orientation. We consider the bundle $\Lambda_n M$ of positive volume forms on M

Recall. Volume form is a scew-symmetic form of maximal order, $Vol = f(x)dx^1 \wedge ... \wedge dx^n$ with $f \neq 0$. "Positive" means that if the basis $\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}$ is positively oriented then $f(x) > 0$.

Positive volume bundle is a locally trivial 1-dimensional bundle over our manifold M with the structure group $(\mathbb{R}_{>0}, \cdot)$. That means in particular that for small neighborhood $U \subset M$ we have an isomorphisms between $\Lambda_n U$ and $\mathbb{R}_{>0} \times U$: there are two natural ways to choose the isomorphism, let us discuss them.

Way A Choose a section in this bundle, i.e., a volume form, the other sections of this bundle can be thought to be positive functions on the manifold (AND IF WE CHANGE COORDINATES THEY TRANSFORM LIKE FUNCTIONS, I.E., DO NOT TRANSFORM AT ALL).

Way B In local coordinates $x = (x^1, ..., x^n)$, we can choose the volume form $dx^1 \wedge ... \wedge dx^n$, the volume form $\Omega = f(x)dx^1 \wedge ... \wedge dx^n$ corresponds to the function $f(x)$. Its transformation rule is different from that of functions: a coordinate change, $x = x(y)$ transforms $f(x)$ to $\det\left(\frac{dx}{dy}\right) f(x(y))$.

$(\Lambda_n)^\alpha M$

Let $\alpha \in \mathbb{R} \setminus \{0\}$. Since $t \to t^\alpha$ is an isomorphism of $(\mathbb{R}_{>0}, \cdot)$, for any 1-dimensional $(\mathbb{R}_{>0}, \cdot)$-bundle its power α is well-defined and is also an one-dimensional bundle. We consider $(\Lambda_n)^\alpha M$. It is an 1-dimensional bundle, so its sections locally can be viewed as functions. Again we have two ways to view the sections as functions:

Way A Choose a volume form Ω, and the corresponding section $\omega = (\Omega)^\alpha$ of $(\Lambda_n)^\alpha M$. Then, the other sections of this bundle can be thought to be positive functions on the manifold.

Way B In local coordinates $x = (x^1, ..., x^n)$, we can choose the section $(dx^1 \wedge ... \wedge dx^n)^\alpha$, then the section $\omega = (f(x) dx^1 \wedge ... \wedge dx^n)^\alpha$ corresponds to the function $(f(x))^\alpha$. Its transformation rule is different from that of functions: a coordinate change, $x = x(y)$ transforms $(f(x))^\alpha$ to $\left(\det\left(\frac{dx}{dy}\right)\right)^\alpha f(x(y))^\alpha$.

Weighted tensors

Def. By a (p,q)-*tensor field of projective weight k* we understand a section of the following bundle:

$$T^{(p,q)} \otimes (\Lambda_n)^{\frac{k}{n+1}} M \quad (\text{notation } := T^{(p,q)} M(k))$$

Way A. If we have a preferred volume form on the manifold, the sections of $T^{(p,q)} M(k)$ can be identified with (p,q)-tensors fields. The identification depends of course on the choice of the volume form.

Way B. If we do not have a preferred volume form on the manifold, in a local coordinate system one can choose $(dx^1 \wedge ... \wedge dx^n)$ as the preferred volume, and still think that sections are "almost" (p,q)-tensors: they are also given by n^{p+q} functions but their transformation rule is slightly different from that for tensors: in addition to the usual transformation rule for tensors one needs to multiply by $\left(\det\left(\frac{dx}{dy}\right)\right)^\alpha$ with $\alpha = \frac{k}{n+1}$.

Weighted tensors in the naive language:

1. If we have a preferred covariantly constant (for a preferred connection in the projective class) volume form, one can identify weighted tensors with usual tensors; in particular the transformation rule and the covariant differentiation will be the same as for usual tensors.

   But: Identification depends on the volume form.
2. Otherwise, they are still very similar to tensors
   - weighted 1-form is locally a n-tuple of functions $(\alpha_1, ..., \alpha_n)$
   - weighted $(0, 2)$-tensor is a $n \times n$-matrix whose components are functions
3. But: If we change a coordinate system, they change by a slightly different transformation rule
4. But: The covariant differentiation (to be explained in the next slides) is given by an other formula than that on for tensors, because we need to covariantly differentiate the volume form as well.
5. Still: Transformation rule and covariant differentiation are compatible – does not matter if we first transform and then differentiate or first differentiate and then transform.

Covariant differentiation of weighted tensor bundles

Fact (e.g. brute force calculations). Suppose (projectively equivalent) connections $\nabla = (\Gamma^i_{jk})$ and $\bar\nabla = (\Gamma^i_{jk})$ are related by the formula

$$\bar\nabla_X V - \nabla_X V = \phi(X)V + \phi(V)X \qquad (**).$$

Then, the covariant derivatives of a volume form $\Omega \in \Gamma(\Lambda_n M)$ in the connections ∇ and $\bar\nabla$ are related by

$$\bar\nabla_X \Omega = \nabla_X \Omega - (n+1)\phi(X)\Omega.$$

In particular, the covariant derivatives of the section $\omega := \left(\Omega^{\frac{k}{n+1}}\right) \in \Gamma((\Lambda_n)^{\frac{k}{n+1}} M)$ are related by

$$\bar\nabla_X \omega = \nabla_X \omega - k\phi(X)\omega. \qquad (2)$$

First example of projectively invariant differential operation

$$\bar\nabla_X\omega = \nabla_X\omega - k\phi(X)\omega. \qquad (2)$$

Let $K \in \Gamma\left(T^{(0,1)}M(-2)\right)$ be an 1-form of projective weight (-2).
We calculate the difference their ∇- and $\bar\nabla$- derivatives assuming

$$\bar\nabla_X V - \nabla_X V = \phi(X)V + \phi(V)X \qquad (**):$$

$$\bar\nabla_X K = \nabla_X K \underbrace{-\phi(X)K - K(X)\phi}_{\text{because of } (**)} + \underbrace{2\phi(X)K}_{\text{because of } (2)} = \nabla_X K + \phi(X)K - K(X)\phi. \quad (3)$$

Theorem. For $(0,1)$-tensors of projective weight (-2) the operation

$$K \mapsto \text{Symmetrization\_Of}(\nabla K) \quad (\text{K1})$$

is projectively invariant: it does not depend on the choice of the affine connection in the projective class.

Proof. Observe in (3) that the difference between $(\bar\nabla_X K)(Y)$ and $(\nabla_X K)(Y)$ is scewsymmetric in X, Y and vanishes after symmetrization.

Remark. In the index notation, the mapping $(K1)$ reads $K_i \mapsto K_{i,j} + K_{j,i}$. The equation $(K1) = 0$ is called projective Killing equation for weighted 1-forms.

Exercises: vectorfields of projective weigth 1

1. Write in indices how $(1,0)$-tensors of projective weight 1 transform under the coordinate change. Write in indices the formula for the covariant derivative of $(1,0)$-tensors of projective weight 1. Use **Way B** for coordinate representation.
2. Prove that for $(1,0)$-tensors of projective weight 1 the operation

   $$\sigma \mapsto \mathit{Trace\_Free\_Part\_Of}\,\nabla(\sigma) = \sigma^i{}_{,j} - \tfrac{1}{n}\sigma^s{}_{,s}\delta^i_j.$$

   is projectively invariant: it does not depend on the choice of the affine connection in the projective class.
3. * Show that a connection $\nabla = (\Gamma^i_{jk})$ is torsionfree iff locally there exists a parallel volume form. Moreover, such volume form is unique up to multiplication by a constant. Hint: preferable direct calculations but if you hate them use that for torsionfree connection we have $[U,V] = \nabla_U V - \nabla_V U$.

   Show by example that globally, on nonsimply connected manifolds, the existence of such a parallel volume form may fail. Hint: Look for example in the class on projectively flat manifolds. Key words are "Hopf foliation".

Theorem. For $(0,1)$-tensors of projective weight (-2) the operation

$$K \mapsto \text{Symmetrization\_Of}(\nabla K)$$

is projectively invariant: it does not depend on the choice of the affine connection in the projective class.

Corollary 1. For symmetric $(0,2)$-tensors of projective weight (-4) the operation

$$K \mapsto \text{Symmetrization\_Of}(\nabla K)$$

is projectively invariant: it does not depend on the choice of the affine connection in the projective class.

Proof. Decompose $(0,2)$ tensors of weight -4 into the sum of symmetric tensor product of $(0,1)$ tensors of weight -2 and apply Corollary 1.

Notation. The equation in Corollary 1 is called projective Killing equation; it will play important role at the end of the lecture.

Metric Projective Geometry: philosophy and goals

One can of course study projective structures without thinking about whether there is a (Levi-Civita connection of a) metric in the projective class.

- Luck of "easy to formulate, hard to prove" results.
- Virtually no "direct" applications in physics

LET US STUDY METRIZABLE PROJECTIVE STRUCTURES, I.E., SUCH THAT THERE EXISTS A METRIC IN THE PROJECTIVE CLASS.

Generic metrizable projective structure has only one, up to a scaling, metric in the projective class (will be proved in exercises). In this case, all geometric questions can be reformulated as questions on this metric.

WE WILL STUDY METRISABLE PROJECTIVE STRUCTURES SUCH THAT THERE EXISTS AT LEAST TWO NONPROPORTIONAL METRICS IN THE PROJECTIVE CLASS.

- Many "easy to formulate, hard to prove" results. Many named problems.
- Many links to other branches of mathematics. Applications in physics.

Next goal: easy to formulate result for today

Theorem (M∼-Topalov 1998). Let (M^2, g) be a two-dimensional closed (compact, no boundary) Riemannian manifold. Assume a metric $\bar g$ is projectively related to g and is nonproportional to g. Then, M^2 has nonnegative Euler characteristic.

I will give an easy proof of this theorem using what we learned today.

I use: projective Killing equation is projectively invariant.

Corollary 2. For $(0, 2)$-tensors of projective weight -4 the operation

$$K \mapsto \text{Symmetrization\_Of}(\nabla K)$$

is projective invariant.

Example. The (Levi-Civita connection of the) metric g does have one nontrivial solution of this equation, namely

$$K = g \otimes (Vol_g)^{\frac{-4}{n+1}}.$$

But the projective Killing equation <u>does not depend</u> on the choice of connection in the projective class.

Thus, any metric in the projective class allows us to construct a solution of the projective Killing equation. Say, if we have another metric $\bar g$ in the same projective class, then

$$\bar K = \bar g \otimes (Vol_{\bar g})^{\frac{-4}{n+1}}.$$

is (still) a solution.

Geometric sense of Killing equations: conservative quantities.

Theorem 4. Suppose K is a solutions of the projective Killing equation. Then, for any metric g in the projective class the tensor field

$$\hat{K} := K \otimes (Vol_g)^{\frac{4}{n+1}}.$$

is a Killing tensor, this means that for any parameterized g-geodesic γ the function

$$t \mapsto I(\gamma(t), \dot\gamma(t)) = K(\dot\gamma(t), \dot\gamma(t))$$

is constant.

Proof. We need to show that

$$\nabla_{\dot\gamma}\left(K(\dot\gamma, \dot\gamma)\right) = 0. \qquad (\star)$$

Because of the definition of geodesic, $\nabla_{\dot\gamma}\dot\gamma = 0$, and $(\star)$ reduces to

$$\nabla K(\dot\gamma, \dot\gamma, \dot\gamma) = 0,$$

which follows from Symmetrization_Of$(\nabla K) = 0$. □

Trivial conservative quantity: energy

Example above. The following section of $T^{(0,2)}M(-4)$

$$K = \bar{g} \otimes (Vol_{\bar g})^{\frac{-4}{n+1}}$$

is a solution of the projective Killing equation for g.

Theorem 4. $\hat{K} := K \otimes (Vol_g)^{\frac{4}{n+1}}$. is a Killing tensor.

If we take the K from the first frame, and use it to construct $\hat{K}$ from the second frame, then we obtain $\hat{K} = g$, which is of course a Killing tensor; the corresponding conservative quantity is the kinetic energy.

Nontrivial conservative quantity, if we have two nonproportional metric in the projective class

Example above. For a metric $\bar g$ in the projective class the following section of $T^{(0,2)}M(-4)$

$$\bar K = \bar g \otimes (Vol_{\bar g})^{\frac{-4}{n+1}}$$

is a solution of the projective Killing equation.

Theorem 4. $\hat K := K \otimes (Vol_g)^{\frac{4}{n+1}}$. is a Killing tensor.

If we take the $\bar K$ from the first frame, and use it to construct $\hat K$ from the second frame, then we obtain $\hat K = \left|\frac{\det g}{\det \bar g}\right|^{\frac{2}{n+1}} \bar g$, which is now nonproportional to the trivial (=always existing) Killing tensor g_{ij}. The corresponding is given by

$$I(x,\xi) = \left|\frac{\det g}{\det \bar g}\right|^{\frac{2}{n+1}} \bar g(\xi,\xi)$$

Historical remark. There are of course direct proofs that I is a conservative quantity, the most classical is possibly due to Painleve 18.. .

Proof of announced theorem

Theorem (M∼-Topalov 1998). Let (M^2, g) be a two-dimensional closed (compact, no boundary) Riemannian manifold. Assume a metric $\bar g$ is projectively related to g and is nonproportional to g. Then, M^2 has nonnegative Euler characteristic.

In dimension 2, the conservative quantity constructed

$$I_0(\xi) := \left|\frac{\det(g)}{\det(\bar g)}\right|^{\frac{2}{3}} \bar g(\xi,\xi).$$

Assume the surface is neither torus nor the sphere. The goal is to show that g and $\bar g$ are proportional.

Because of topology, there exists x_0 such that $g_{|x_0} = \text{const} \cdot \bar g_{|x_0}$. W.l.o.g. we assume const $= 1$. We assume $g_{|x_1} \neq \bar g_{|x_1}$ and find a contradiction (will be done on the blackboard).

Exercises for the exercise section

1. Prove: if two 2-dimensional metrics at least one of which is complete are projectively equivalent and proportional in more than 4 points than they are proportional everywhere with a constant coefficient.

   Hint: first show that the coefficient of proportionality should necessary be a constant.

2. Prove the Weyl Theorem 1924: it two metrics of any dimension ≥ 2 are projectively equivalent everywhere and conformally equivalent on an open nonempty subset then they are proportional with a constant coefficient.

   Hint: Act as in exercise above. Replace the argument that intersection of different quadrics contains no more than 4 points by the following: intersections of quadrics is an algebraic subset of and any algebraic subset of $\mathbb{R}^n$ containing an open nonnempty subset is the whole $\mathbb{R}^n$.

3. * Prove that in all dimensions ≥ 2 the set of the points where the metrics are proportional is a totally geodesic submanifold.
4. * Prove (at least in dimension 2) that a generic in the C^2-sense metric does not admit nonproportional projective equivalence

   Hint. The general ideas is the same as everywhere above; you need to consider conservative quantities. If brainstorm does not work, read §2.1 of the paper of Kruglikov and Matveev "The geodesic flow of a generic metric does not admit nontrivial integrals polynomial in momenta".

Lecture 2

Plan

- Metrization equation
- Local normal forms of projectively related Riemannian metrics
- Problems of Lie and their solution
- What allowed us to solve the problems of Lie?

Two more projectively invariant differential operators

Theorem. For $(1,0)$-tensors of projective weight 1 the operation

$$\sigma \mapsto \textit{Trace\_Free\_Part\_Of}\,\nabla(\sigma) = \sigma^i{}_{,j} - \tfrac{1}{n}\sigma^s{}_{,s}\delta^i_j.$$

is projectively invariant: it does not depend on the choice of the affine connection in the projective class.

Proof was an exercise for the exercise section and is essentially the same as for Corollary 1.

Corollary 3. For symmetric $(2,0)$-tensors of projective weight 2 the operation

$$\sigma^{ij} \mapsto \sigma^{ij}{}_{,k} - \tfrac{1}{n+1}(\sigma^{is}{}_{,s}\delta^j_k + \sigma^{js}{}_{,s}\delta^i_k) \qquad (4)$$

is projectively invariant: it does not depend on the choice of the affine connection in the projective class.

Proof. Decompose $(2,0)$ tensors of weight 2 into the sum of symmetric tensor products of $(0,1)$ tensors of weight 1 and apply Corollary 2

Remark. In the index-free notation the operation (4) reads

$$\sigma \mapsto \text{Trace\_Free\_Part\_Of}\,(\nabla\sigma)\,,$$

though $\nabla\sigma$ is a (2,1)-(weighted)-tensor and trace is a (weighted) vector.

Geometric importance of the operator $\sigma \mapsto$ Trace_Free_Part_Of $(\nabla\sigma)$.

(Metrization) Theorem 3 (Eastwood-M$\sim$ 2006). Suppose the Levi-Civita connection of a metric g lies in a projective class $[\nabla]$. Then, $\sigma^{ij} := g^{ij} \otimes (Vol_g)^{\frac{2}{n+1}}$ is a solution of

$$\text{Trace\_Free\_Part\_Of}\,(\nabla\sigma) = 0. \qquad (5)$$

Moreover, for every solution of the equation (5) such that $\det(\sigma) \neq 0$ there exists a metric whose Levi-Civita connection lies in the projective class.

Proof in the direction $\Rightarrow$. We assume that $\nabla^g \in [\nabla]$. Since our equation is projectively invariant, we may assume that we work in the connection ∇^g. In this connection the metric and therefore all objects constructed by the metric are parallel so $\nabla^g(\sigma) = 0$ which of course implies (5)

(Metrization) Theorem 3. Suppose the Levi-Civita connection of a metric g lies in a projective class $[\nabla]$. Then, $\sigma^{ij} := g^{ij} \otimes (Vol_g)^{\frac{2}{n+1}}$ is a solution of

$$\text{Trace\_Free\_Part\_Of}\,(\nabla\sigma) = 0. \qquad (5)$$

Moreover, for every solution of the equation (5) such that $\det(\sigma) \neq 0$ there exists a metric whose Levi-Civita connection lies in the projective class.

Proof in the direction $\Leftarrow$ is for the exercise session, let me give some hints. Observe that though equation

$$\sigma^{ij} \mapsto \sigma^{ij}{}_{,k} - \frac{1}{n+1}\left(\sigma^{is}{}_{,s}\delta^j_k + \sigma^{js}{}_{,s}\delta^i_k\right)$$

does not depend on the choice of a connection in the projective class, the part of it marked by *blue color* does depend. Find out how it depends and prove that there exists a connection in the projective class such that the *blue* part is zero, show then that this connections preserves a metric such that σ is obtained by the metric by the formula in Theorem 3.

Relation between (nondegenerate) solutions σ of metrization equations and metrics in coordinates

Let us work in a coordinate system and choose $dx^1 \wedge ... \wedge dx^n$ as a volume form.

- If we have a metric g_{ij}, then the corresponding solution of the metrization equation is given by

$$\sigma^{ij} := \left(g^{ij} \otimes (Vol_g)^{\frac{2}{n+1}}\right) = g^{ij}|\det g|^{\frac{1}{n+1}}.$$

- For a solution $\sigma = \sigma^{ij}$ of the metrisation equation such that its determinant in not zero, the corresponding metric is given by

$$g^{ij} := |\det(\sigma)|\sigma^{ij}.$$

Metrization equations in dimension 2 in coordinates:

Theorem 3 (Metrization equations in all dimensions:) $\text{Trace\_Free\_Part\_Of}(\nabla\sigma) = 0.$ (5)

As we remember from Lecture 1, in dimension 2 the four functions K_0, K_1, K_2, K_3 (which are the coefficients of the equation) (essentially R. Liouville 1889)

$$y'' = \underbrace{-\Gamma^2_{11}}_{K_0} + \underbrace{(\Gamma^1_{11} - 2\Gamma^2_{12})}_{K_1} y' + \underbrace{(2\Gamma^1_{12} - \Gamma^2_{22})}_{K_2} y'^2 + \underbrace{\Gamma^1_{22}}_{K_3} y'^3.$$

encode the projective class of the connection Γ^i_{jk}.

In this setting, the metrization equations in the following system of 4 PDE on three unknown functions:

$$\left\{\begin{array}{rcl} {\sigma^{22}}_x - \frac{2}{3} K_1 \sigma^{22} - 2 K_0 \sigma^{12} & = & 0 \\ {\sigma^{22}}_y - 2 {\sigma^{12}}_x - \frac{4}{3} K_2 \sigma^{22} - \frac{2}{3} K_1 \sigma^{12} + 2 K_0 \sigma^{11} & = & 0 \\ -2 {\sigma^{12}}_y + {\sigma^{11}}_x - 2 K_3 \sigma^{22} + \frac{2}{3} K_2 \sigma^{12} + \frac{4}{3} K_1 \sigma^{11} & = & 0 \\ {\sigma^{11}}_y + 2 K_3 \sigma^{12} + \frac{2}{3} K_2 \sigma^{11} & = & 0 \end{array}\right.$$

Corollary. Generic projective structure is not metrizable.
Explanation (formal proof in Bryant et al 2009 or Kruglikov-M∼2015). The system is overdertermined: 4 equations on three unknown functions, and generic overdetermined systems have no solution

Our first goal is to prove the Dini's Theorem 1869

Local normal form question (Beltrami 1865): Given two projectively related metric, how do they look in "the best" coordinate system (near a generic point)? How unique is such best coordinate system?

Theorem (Dini 1869). Let g and $\bar g$ are projectively related 2 dim Riemannian metrics. Then, in a neighborhood of almost every point there exists a coordinate system such that in this coordinate system the metrics are

$$g = \begin{pmatrix} X(x) - Y(y) & \\ & X(x) - Y(y) \end{pmatrix}$$

$$\bar g = \begin{pmatrix} \frac{X(x)-Y(y)}{X(x)Y(y)^2} & \\ & \frac{X(x)-Y(y)}{X(x)^2 Y(y)} \end{pmatrix}$$

The coordinates are unique modulo $(x, y) \mapsto (\pm x + b, \pm y + d)$.

Remark. The answer in higher dimensions is also known (Levi-Civita) In other signatures the answer to the Beltrami questions is also known (Darboux/Lie for dim 2, Bolsinov-Matveev 2013 for all dimensions).

Rem. In the 2 dim case of splitted signature, there are two more cases: when $g^{-1}\bar g$ has complex eiganvalues, and when $g^{-1}\bar g$ has Jordan block.

Proof: coordinates such that g and $\bar{g}$ are diagonal

Such coordinates exist near every generic points:

Indeed, at the points where g is not proportional to $\bar{g}$ the (1,1)-tensor $g^{-1}\bar{g} = g^{is}\bar{g}_{js}$ has two different eigenvalues. We consider the coordinate system (x, y) such that $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are eigenvectors.

Since the eigenvectors are orthogonal w.r.t. g and w.r.t. $\bar{g}$, in this coordinates the metrics are diagonal.

Plugging diagonal $\sigma, \bar{\sigma}$ in the metrization theorem

Thus, we may assume that in the coordinate system (x, y) the metrics are diagonal and theirfore the corresponding solutions of the metrization equation $\sigma = \left(g^{ij} \otimes (Vol_g)^{\frac{2}{n+1}}\right) = g^{ij}(\det g)^{\frac{1}{n+1}}$,

$\bar{\sigma} = \left(\bar{g}^{ij} \otimes (Vol_{\bar{g}})^{\frac{2}{n+1}}\right) = \bar{g}^{ij}(\det \bar{g})^{\frac{1}{n+1}}$ are diagonal.

Consider the (1,1)-tensor field $A = \bar{\sigma}(\sigma)^{-1} = \bar{\sigma}^{is}\sigma_{js}$, it is also diagonal:

$$\sigma = \begin{pmatrix} \sigma^{11} & \\ & \sigma^{22} \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix}, \quad \bar{\sigma} = \begin{pmatrix} A_1\sigma^{11} & \\ & A_2\sigma^{22} \end{pmatrix}.$$

Let us now plug these σ and $\bar{\sigma}$ in the metrization theorem whose two-dimensional version is in Lecture 2:

$$\begin{aligned} {\sigma^{22}}_x - \tfrac{2}{3} K_1 \sigma^{22} - 2 K_0 \sigma^{12} &= 0 \\ {\sigma^{22}}_y - 2 {\sigma^{12}}_x - \tfrac{4}{3} K_2 \sigma^{22} - \tfrac{2}{3} K_1 \sigma^{12} + 2 K_0 \sigma^{11} &= 0 \\ -2 {\sigma^{12}}_y + {\sigma^{11}}_x - 2 K_3 \sigma^{22} + \tfrac{2}{3} K_2 \sigma^{12} + \tfrac{4}{3} K_1 \sigma^{11} &= 0 \\ {\sigma^{11}}_y + 2 K_3 \sigma^{12} + \tfrac{2}{3} K_2 \sigma^{11} &= 0 \end{aligned}$$

Elementary tricks solve the system

$$\begin{array}{rcl|rcl} {\sigma^{22}}_x - \frac{2}{3} K_1 \sigma^{22} &=& 0 & A_2{\sigma^{22}}_x + (A_2)_x\sigma^{22} - \frac{2}{3} K_1 A_2\sigma^{22} &=& 0 \\ {\sigma^{22}}_y - \frac{4}{3} K_2 \sigma^{22} + 2 K_0 \sigma^{11} &=& 0 & A_2{\sigma^{22}}_y + (A_2)_y\sigma^{22} - \frac{4}{3} K_2 A_2\sigma^{22} + 2 K_0 A_1\sigma^{11} &=& 0 \\ {\sigma^{11}}_x - 2 K_3 \sigma^{22} + \frac{4}{3} K_1 \sigma^{11} &=& 0 & A_1{\sigma^{11}}_x + (A_1)_x\sigma^{11} - 2 K_3 A_2\sigma^{22} + \frac{4}{3} K_1 A_1\sigma^{11} &=& 0 \\ {\sigma^{11}}_y + \frac{2}{3} K_2 \sigma^{11} &=& 0 & A_1{\sigma^{11}}_y + (A_1)_y\sigma^{22} + \frac{2}{3} K_2 A_1\sigma^{11} &=& 0 \end{array}$$

Message: systems of PDE with more equations are as a rule easier to solve than that with less equations

How we proceed: Solve the first 4 questions with respect to $K_0, ..., K_3$ and substitute the result in the last 4 equations. One obtains the equations

$$\left(\begin{array}{rcl} (A_1)_y &=& 0 \\ (A_2)_x &=& 0 \\ ((A_1 - A_2)\sigma^{11}(\sigma^{22})^2)_x &=& 0 \\ ((A_1 - A_2)\sigma^{22}(\sigma^{11})^2)_y &=& 0. \end{array}\right) \text{ implying } \left(\begin{array}{rcl} A_1 &=& X(x) \\ A_2 &=& Y(y) \\ (X(x) - Y(y))\sigma^{11}(\sigma^{22})^2 &=& \frac{1}{Y_1(y)} \\ (X(x) - Y(y))\sigma^{22}(\sigma^{11})^2 &=& \frac{1}{X_1(x)}. \end{array}\right)$$

$$\left(\begin{array}{rcl} A_1 &=& X(x) \\ A_2 &=& Y(y) \\ (X(x) - Y(y))\sigma^{11}(\sigma^{22})^2 &=& \frac{1}{Y_1(y)} \\ (X(x) - Y(y))\sigma^{22}(\sigma^{11})^2 &=& \frac{1}{X_1(x)}. \end{array}\right)$$

Observe now, because of the relation $g^{ij} = |\det(\sigma)|\sigma$ and because of the matrices $\sigma, \bar\sigma$ are diagonal, we have $\sigma^{11}(\sigma^{22})^2 = g^{22}$ and $\sigma^{22}(\sigma^{11})^2 = g^{11}$. Thus, we obtain that

$$g = (X - Y)(X_1 dx^2 + Y_1 dy^2) \text{ and } A = diag(X, Y).$$

By a coordinate change $x = x(x_{new}), y = y(y_{new})$, one can "hide" X_1 and Y_1 in dx^2 and dy^2 and obtain the formulas of Dini □

Projective transformations

Def. Projective transformation of a projective structure $[\Gamma]$ is a (local) diffeomorphism that preserves $[\Gamma]$.

Geometric (equivalent) definition. Projective transformations are diffeomorphisms that send geodesics to geodesics.

Example. Affine transformations from 1st year linear algebra course (i.e., $x \mapsto Ax + b$ with nondegenerate matrix A) are projective transformations of the flat (i.e., when $\Gamma^i_{jk} \equiv 0$) projective structure.

Example. Projective transformations from linear algebra are projective transformations of the flat projective structure.

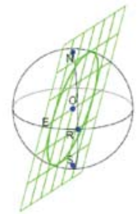
Def. A vector field is projective w.r.t. $[\Gamma]$, if its (local) flow acts by projective transformations.

Example. A Killing vector field of a metric is projective w.r.t. the projective structure of the metric.

Beltrami example: projective algebra of the round sphere is $sl(n+1)$.

We consider the standard $S^n \subset R^{n+1}$ with the induced metric.

Fact. Geodesics of the sphere are the great circles, that are the intersections of the 2-planes containing the center of the sphere with the sphere.

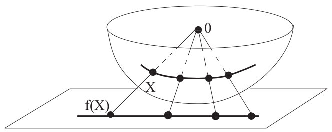


Beltrami (1865) observed:
For every $A \in SL(n+1) \xrightarrow{\text{we construct}} a : S^n \to S^n, a(x) := \frac{A(x)}{|A(x)|}$

- a is a diffeomorphism
- a takes great circles (geodesics) to great circles (geodesics)
- a is an isometry iff $A \in O(n+1)$.

Thus, $Sl(n+1)$ acts by projective transformations on S^n. Its stabilizator is discrete and therefore the algebra of projective vector fields is $sl(n+1)$; in dimension $n = 2$ it has dimension $(n+1)^2 - 1 = 8$.

Example of Lagrange 1789



Radial projection $f : S^2 \to \mathbb{R}^2$ takes geodesics of the sphere to geodesics of the plane, because geodesics on sphere/plane are intersection of plains containing 0 with the sphere/plane.

Thus, the projective structure of the plane is the same as that of the sphere and also has 8-dimensional projective algebra.
Everything survives to all dimensions and all signatures and for negative curvature

"Nice" result of todays lecture: Problems of Lie

Lie 1882:

Problem I: *Es wird verlangt, die Form des Bogenelementes einer jeden Fläche zu bestimmen, deren geodätische Kurven eine infinitesimale Transformation gestatten.*

English translation:

Describe all 2 dim metrics admitting

- **Problem I: one projective vector field**
- **Problem II: many projective vector fields**

Lie 1882:

Problem II: *Man soll die Form des Bogenelementes einer jeden Fläche bestimmen, deren geodätische Kurven mehrere infinitesimale Transformationen gestatten.*

Both problems are local, in a neighborhood of a generic point

Solution of the 2nd Lie Problem

Theorem (Bryant, Manno, M∼ 2007) *If a two-dimensional metric g of nonconstant curvature has at least 2 projective vector fields such that they are linear independent at the point p, then there exist coordinates x, y in a neighborhood of p such that the metrics are as follows.*

1. $\varepsilon_1 e^{(b+2)x}dx^2 + \varepsilon_2 b e^{bx}dy^2$, *where* $b \in \mathbb{R} \setminus \{-2, 0, 1\}$ *and* $\varepsilon_i \in \{-1, 1\}$
2. $a\left(\varepsilon_1 \frac{e^{(b+2)x}dx^2}{(e^{bx}+\varepsilon_2)^2} + \frac{e^{bx}dy^2}{e^{bx}+\varepsilon_2}\right)$, *where* $a \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R} \setminus \{-2, 0, 1\}$ *and* $\varepsilon_i \in \{-1, 1\}$
3. $a\left(\frac{e^{2x}dx^2}{x^2} + \varepsilon \frac{dy^2}{x}\right)$, *where* $a \in \mathbb{R} \setminus \{0\}$, *and* $\varepsilon \in \{-1, 1\}$
4. $\varepsilon_1 e^{3x}dx^2 + \varepsilon_2 e^{x}dy^2$, *where* $\varepsilon_i \in \{-1, 1\}$,
5. $a\left(\frac{e^{3x}dx^2}{(e^x+\varepsilon_2)^2} + \frac{\varepsilon_1 e^x dy^2}{(e^x+\varepsilon_2)}\right)$, *where* $a \in \mathbb{R} \setminus \{0\}$, $\varepsilon_i \in \{-1, 1\}$,
6. $a\left(\frac{dx^2}{(cx+2x^2+\varepsilon_2)^2 x} + \varepsilon_1 \frac{x dy^2}{cx+2x^2+\varepsilon_2}\right)$, *where* $a > 0$, $\varepsilon_i \in \{-1, 1\}$, $c \in \mathbb{R}$.

Theorem (M∼ 2008): *Let v be a projective vector field on $(M^2, \bar{g})$. Assume the restriction of $\bar{g}$ to no neighborhood has an infinitesimal homothety. Then, there exists a coordinate system in a neighborhood of almost every point such that certain metric g geodesically equivalent to $\bar{g}$ is given by*

1. $ds_g^2 = (X(x) - Y(y))(X_1(x)dx^2 + Y_1(y)dy^2)$, $v = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, where
   - 1.1 $X(x) = \frac{1}{x}$, $Y(y) = \frac{1}{y}$, $X_1(x) = C_1 \cdot \frac{e^{-3x}}{x}$, $Y_1(y) = \frac{e^{-3y}}{y}$.
   - 1.2 $X(x) = \tan(x)$, $Y(y) = \tan(y)$, $X_1(x) = C_1 \cdot \frac{e^{-3\lambda x}}{\cos(x)}$, $Y_1(y) = \frac{e^{-3\lambda y}}{\cos(y)}$.
   - 1.3 $X(x) = C_1 \cdot e^{\nu x}$, $Y(y) = e^{\nu y}$, $X_1(x) = e^{2x}$, $Y_1(y) = \pm e^{2y}$.
2. $ds_g^2 = (Y(y) + x)dxdy$, $v = v_1(x,y)\frac{\partial}{\partial x} + v_2(y)\frac{\partial}{\partial y}$, where
   - 2.1 $Y = e^{\frac{3}{2y}} \cdot \frac{\sqrt{y}}{y-3} + \int_{y_0}^{y} e^{\frac{3}{2\xi}} \cdot \frac{\sqrt{\xi}}{(\xi-3)^2}d\xi$,
     $v_1 = \frac{y-3}{2}\left(x + \int_{y_0}^{y} e^{\frac{3}{2\xi}} \cdot \frac{\sqrt{\xi}}{(\xi-3)^2}d\xi\right)$, $v_2 = y^2$.
   - 2.2 $Y = e^{-\frac{3}{2}\lambda \arctan(y)} \cdot \frac{\sqrt[4]{y^2+1}}{y-3\lambda} + \int_{y_0}^{y} e^{-\frac{3}{2}\lambda \arctan(\xi)} \cdot \frac{\sqrt[4]{\xi^2+1}}{(\xi-3\lambda)^2}d\xi$,
     $v_1 = \frac{y-3\lambda}{2}\left(x + \int_{y_0}^{y} e^{-\frac{3}{2}\lambda \arctan(\xi)} \cdot \frac{\sqrt[4]{\xi^2+1}}{(\xi-3\lambda)^2}d\xi\right)$, $v_2 = y^2 + 1$.
   - 2.3 $Y(y) = y^{\nu}$, $v_1(x,y) = \nu x$, $v_2 = y$.

Why the problems of Lie were not solved before? What know-how allowed us to solve it?

- Many people tried (including Lie and his students)
- One immediately reformulates the problem as a (quasilinear) 2ND ORDER system of PDE on the components of metric and of the vector field; the system is to hard to solve by hands.
- OUR NEW VIEWPOINT ON THE PROBLEM WHICH ALLOWED TO SOLVE IT WAS TO USE PROJECTIVELY-INVARIANT OBJECTS.
- This allowed to reduced the PDE-reformulation to MORE equations of the 1ST order which can be solved by hands.

How we solve the problems of Lie: first observation:

We had two projectively invariant equations: Killing equations and metrization equations: let us compare them in dimension 2:

Metrization equation in dimension 2:

$$\begin{aligned} {\sigma^{22}}_x - \tfrac{2}{3} K_1 \sigma^{22} - 2 K_0 \sigma^{12} &= 0\\ {\sigma^{22}}_y - 2 {\sigma^{12}}_x - \tfrac{4}{3} K_2 \sigma^{22} - \tfrac{2}{3} K_1 \sigma^{12} + 2 K_0 \sigma^{11} &= 0\\ -2 {\sigma^{12}}_y + {\sigma^{11}}_x - 2 K_3 \sigma^{22} + \tfrac{2}{3} K_2 \sigma^{12} + \tfrac{4}{3} K_1 \sigma^{11} &= 0\\ {\sigma^{11}}_y + 2 K_3 \sigma^{12} + \tfrac{2}{3} K_2 \sigma^{11} &= 0 \end{aligned}$$

Killing equation in dimension 2:

$$\begin{aligned} a_{11x} - \tfrac{2}{3} K_1 a_{11} + 2 K_0 a_{12} &= 0\\ a_{11y} + 2 a_{12x} - \tfrac{4}{3} K_2 a_{11} + \tfrac{2}{3} K_1 a_{12} + 2 K_0 a_{22} &= 0\\ 2 a_{12y} + a_{22x} - 2 K_3 a_{11} - \tfrac{2}{3} K_2 a_{12} + \tfrac{4}{3} K_1 a_{22} &= 0\\ a_{22y} - 2 K_3 a_{12} + \tfrac{2}{3} K_2 a_{22} &= 0 \end{aligned}$$

We see that the equations coincide after renaming the variables:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = Comatrix\left(\begin{pmatrix} \sigma^{11} & \sigma^{12} \\ \sigma^{12} & \sigma^{22} \end{pmatrix}\right) = \begin{pmatrix} \sigma^{22} & -\sigma^{12} \\ -\sigma^{12} & \sigma^{11} \end{pmatrix}. \qquad (*)$$

Remark. The operation of taking the comatrix is a "geometric" operation: it does not depend on the coordinate system, it is invertible, and in dimension 2 it gives a linear bijection between $(2,0)$-tensors of projective weight 2 and $(0,2)$-sections of projective weight -4.

We just have proved the following theorem:

Theorem. In dimension 2, solutions of metrization equations are in one-to-one correspondence to the solutions of projective Killing equations (Killing tensors are assumed symmetric), the correspondence is given in coordinates by

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = Comatrix\left(\begin{pmatrix} \sigma^{11} & \sigma^{12} \\ \sigma^{12} & \sigma^{22} \end{pmatrix}\right). \qquad (*)$$

Corollary. Suppose there exists a solution σ^{ij} of the metrisation equation such that a_{ij} is degenerate but nonzero (in every point). Then, there exists a (projective) Killing 1-form, and in the case we have a metric in the projective class, a Killing vector field.

Proof. In dimension 2, comatrix of a nonzero degenerate matrix is a nozero degenerate matrix. It has therefore rank 1 and degenerate nonzero projective Killing tensor has the form $a = \pm K \otimes K$ for some 1-form K. This 1-form is Killing: one can see it from equations but let us see it geometrically in metric situation which is sufficient for our goals.

The corresponding conservative quantity is $K(\xi) = \sqrt{\pm a(\xi,\xi)}$. Since the function $a(\xi,\xi)$ is preserved along geodesics, the function $K(\xi)$ is also preserved along geodesics, so that K is a Killing one form (and after raising the index we obtain a Killing vector field).

We may assume that our metrics do not have Killing vector fields

Corollary. Suppose there exists a solution σ^{ij} of the metrisation equation such that a_{ij} is degenerate but nonzero (in every point). Then, there exists a (projective) Killing 1-form, and in the case we have a metric in the projective class, a Killing vector field.

Killing vector field is automatically projective, so without loss of generality in the solution of the first Lie problem we assume nonexistence of a Killing vector field and therefore we assume that all nonzero solutions of the metrization equation we meet are nondegenerate.

Lie derivative w.r.t. projective vector field sends solutions of the metrization equations to solutions

Claim. Let v be projective for $[\Gamma]$ and suppose σ is a solution of the metrization equation. Then, $\mathcal{L}_v\sigma$ is also a solution of the metrization equation.

Proof in the 2 dim case. Everything is coordinate-invariant, so w.l.o.g. we can work in coordinates (x,y) such that $v=\frac{\partial}{\partial x}$. In this coordinates, the coefficients $K_0,\ldots,K_3$ do not depend on x, so the coefficients of the metrization equation

$$\begin{array}{rcl}
{\sigma^{22}}_x - \frac{2}{3}K_1\sigma^{22} - 2K_0\sigma^{12} &=& 0\\
{\sigma^{22}}_y - 2{\sigma^{12}}_x - \frac{4}{3}K_2\sigma^{22} - \frac{2}{3}K_1\sigma^{12} + 2K_0\sigma^{11} &=& 0\\
-2{\sigma^{12}}_y + {\sigma^{11}}_x - 2K_3\sigma^{22} + \frac{2}{3}K_2\sigma^{12} + \frac{4}{3}K_1\sigma^{11} &=& 0\\
{\sigma^{11}}_y + 2K_3\sigma^{12} + \frac{2}{3}K_2\sigma^{11} &=& 0
\end{array}$$

do not depend on x as well. Then, for any solution σ^{ij} the $\frac{\partial}{\partial x}$-Lie derivative (will be explained on trivial language on the next slide), which is simply $\frac{\partial}{\partial x}$ is also a solution □

Remark. The proof actually works in other dimensions as well – we simply need to observe that in coordinates $(x^1,\ldots,x^n)$ such that $v=\frac{\partial}{\partial x^1}$ the coefficients of the metrization equation does not depend on x^1.

PDE background of the trick

We consider the following linear system of PDE:

$$\sum_{k,i} c_j^{i,k}\frac{\partial u_i}{\partial x_k}+\sum_i c_j^i u_i=0\ ,\quad j=1,\ldots,m. \tag{1}$$

Here $(u_1,\ldots,u_\ell)$ are the unknown functions to find, the coefficients $c_j^{i,k}$ and c_j^i are functions thought to be known, everything lives in a small neighborhood $W\subset\mathbb{R}^n$ and is at least as smooth as I need in the proofs.

Fact (1st year calculus). Assume the coefficients $c_j^{i,k}$ and c_j^i are independent of x_1. Then, for any solution $(u_1,\ldots,u_\ell)$ of (1), the tuple $\left(\frac{\partial}{\partial x_1}u_1,\ldots,\frac{\partial}{\partial x_1}u_\ell\right)$ is also a solution.

Proof. We differentiate the equations (1) and interchange the partial derivatives to obtain

$$\frac{\partial}{\partial x_1}\left(\sum_{k,i}c_j^{i,k}\frac{\partial u_i}{\partial x_k}+\sum_i c_j^i u_i\right)=\sum_{k,i}c_j^{i,k}\frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial x_1}u_i\right)+\sum_i c_j^i\left(\frac{\partial}{\partial x_1}u_i\right)=0\,.$$

The action of the Lie derivative of a projective vector field on the space of solutions of the metrization equation

Notation. We denote the space of solutions of metrization equation by $Sol([\Gamma])$.

Fact (Liouville 1889 in dim 2, Sinjukov 1959 in dim n), will be in the exercise section: $\dim(Sol([\Gamma])) \le \frac{(n+1)(n+2)}{2} < \infty$.

Consider the linear mapping $\mathcal{L}_v : Sol([\Gamma]) \to Sol([\Gamma])$. Well-defined because by Claim above Lie derivative of a solution is a solution.

Fact from linear algebra. If $\dim(Sol([\Gamma])) \ge 2$, there exists a two-dimensional invariant subspace of L_v, we will work with this subspace and forget the rest.

By linear algebra, there exists a basis $\sigma, \bar\sigma \in Sol([\Gamma])$ such that in this basis the matrix of $\mathcal{L}_v$ is given by the following (real) Jordan normal form.

$$\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}, \begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

As we explained above, w.l.o.g. we may assume that σ and $\bar\sigma$ are nondegenerate; then they correspond to some metrics.

How to get more equations of less order

We have 3 possibilities for the action of $\mathcal{L}_v$ depending on the Jordan form of $\mathcal{L}_v : Sol \to Sol$ and also 3 possibilities for local normal forms (you have seen the one, when one of the metrics is Riemannian. There are two more: in the "complex" case the freedom is a choice of a holomprphic function, and in the "Jordan" case the freedom is one function of one variable.
Thus, there are 9 different cases

Take one of this cases, then the metrics are given by explicit formulas with freedom (=unknown functions in our system of equations) being choice of two functions of one variable, one holomorphic function, or one function of one variable. Two more unknown functions are the components $v^1(x,y), v^2(x,y)$ of the projective vector field.
Let us count the equations: they are

$$\mathcal{L}_v\sigma = a\sigma + b\bar\sigma \quad \text{3 equations of first order}$$
$$\mathcal{L}_v\bar\sigma = c\sigma + d\bar\sigma \quad \text{3 equations of first order}$$

Here $\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ is one of the jordan normal forms above.
Thus, we have 6 equations of the first order on 2 functions $v^1(x,y), v^2(x,y)$ of two variables, and, depending on the case, on one or two functions of one variables or on a holomorphic function. We see that the number derivatives of the unknown functions are no greater than the number of equations so one can solve the equations w.r.t. derivatives and such systems are easy to solve.

As example consider the simplest case: the matrix of $\mathcal{L}_v$ is $\begin{pmatrix}\lambda_1 & \\ & \lambda_2\end{pmatrix}$, and <u>assume in addition</u> that $\sigma, \bar\sigma \in (Sol([\Gamma]))$ corresponds to <u>Riemannian</u> metrics. Then, by the Dini Theorem the metrics g and $\bar g$ corresponding to σ and $\bar\sigma$ are given by

$$g = \begin{pmatrix} X(x)-Y(y) & \\ & X(x)-Y(y)\end{pmatrix}$$
$$\bar g = \begin{pmatrix} \frac{X(x)-Y(y)}{X(x)Y(y)^2} & \\ & \frac{X(x)-Y(y)}{X(x)^2Y(y)}\end{pmatrix}$$

Observe that the projective vector field v preserves the pencil of the solutions $\alpha\sigma + \beta\bar\sigma$, and therefore any object constructed by these solutions, in particular the lines of the coordinates (x,y). Then, the vector field $v = (v^1(x), v^2(y))$. Now, from $\mathcal{L}_v\sigma = \lambda\sigma$ it follows that v preserves the conformal structure of the metric g, so it is a holomorphic vector field. Then, it is constant or linear, i.e., up to a coordinate change and factor it is either

$$v = (x,y) \quad \text{or} \quad v = (\mathrm{const}_1, \mathrm{const}_2).$$

In both cases the flow is given by precise formulas, which after some work give formulas for $X(x)$ and $Y(y)$ from case 1.1. of Theorem.

Few words about the solution of the second problem of Lie

Recall: Difference between the first and the second problem: in the 1st problem we look for metrics with one projective vector field, and in the 2nd problem with many.

- One should of course check whether the metrics we obtained do not have another projective vector field.
  - This is an algorithmically doable problem assuming we can differentiate and arithmetic operations.
  - The algorithm is essentially due to Lie and is build in Maple and since the metrics in Theorem are explicit Maple can work with them and gives an answer.
- Then, the only additional problem to solve is to omit the assumption that their exists no Killing vector field. But if there exists a Killing vector field, one can use again projective invariance of the Killing equation (I do not go into details at this point)

Historical Remark. In this lecture I first solved the 1st problem, and then used it in the 2nd problems of Lie, historically first the 2nd problem was solved (Bryant, Manno, M$\sim$ 2006) and then the 1st (M$\sim$ 2008); but the solution of the 2nd without having the 1st is computationally quite hard.

What Lie did not know? Why he did not solved his problems himself?

Ingredients of the proof.

- Local normal forms of projectively equivalent metrics? **Lie knew it**
- Linear algebra? **Lie understood it much better than most of us.**
- Quite big calculations (9 cases etc)? **Read any paper of Lie and see how good he was in calculations.**

• He did not know the projective invariance of the metrization equation!!!
• Message of this lecture: projective invariance is important!!!
• In the next lecture we will construct another type of projectively invariant objects and prove a classical conjecture with their help

Exercises:

1. Dini Theorem and Lie problem on the torus
   - Show that for two projectively equivalent 2-dimensional metrics near the points where they are not proporional one can canonically construct two commutative eigenvectorfields such that in the corresponding coordinate system the metrics are given by Dini's formulas. Hint: observe that in the Dini coordinate system the functions $X(x)$ and $Y(y)$ can be defined in the terms of eigenvalues of $g^{-1}\bar{g}$; and then use them to "normalise" the eigenvectors.
   - Use this to describe all pairs of projectively equivalent metrics on the torus Hint: In the exercise secton yesterday we have shown that on complete manifold the number of the points where the metrics are proportional is at most two. Use this to show that on the torus projectively equivalent metrics are proportional at no points.
   - Show that on the 2- torus equipped with a nonflat Riemannian metric any projective vector field is Killing (as will be explained tomorrow, this is a special case of the Lichnerowicz conjecture).

     Hint: We know the formulas for the metrics admitting projective vector field; see that solution of the Lie problem. You need to show that in view of the previous exercise the formulas are defined on the whole torus and then the evolution of functions $X(x)$ and $Y(y)$ leads to a contradiction.

(3) Lie algebra of projective vector fields
   - Show that in dimension 2 and assuming that the metric is not of constant curvature situation any projective vector field V and any Killing vector field K satisfy the relation $[K, V] = \mathrm{const} \cdot K$.

     Hint: Use that there is only one, up to a constant coefficient, Killing vector field and apply the PDE-trick from the lecture.
   - (*) Show without using theorem about the solution of the 2nd problem of Lie that const from $[K, V] = \mathrm{const} \cdot K$ can not be zero for metrics of nonconstant curvature

Lecture 3

Plan

- Tensor invariants of the projective structure: Weyl and Liouville tensors
- Proof of Lichnerowicz conjecture

What are tensor invariants?

Tensor invariants of a projective structure are tensor fields canonically constructed by an affine connection in the projective structure such that they do not depend on the choice of affine connection within this projective structure.

$$\bar\Gamma^i_{jk} = \Gamma^i_{jk} + \phi_k \delta^i_j + \phi_j \delta^i_k. \qquad (*)$$

Example. Curvature and Ricci tensors are NOT tensor invariants. Indeed, if we replace a connection Γ by the connection $\bar\Gamma$, then the direct calculations using the straightforward formula

$$R^m{}_{ikp} = \partial_k \Gamma^m{}_{ip} - \partial_p \Gamma^m{}_{ik} + \Gamma^a{}_{ip}\Gamma^m{}_{ak} - \Gamma^a{}_{ik}\Gamma^m{}_{ap}$$

give us the following relation between the curvature tensors of Γ and $\bar\Gamma$:

$$\bar R^h{}_{ijk} = R^h{}_{ijk} + (\phi_{j,k} - \phi_{k,j})\delta^h{}_i + \delta^h{}_k\left(\phi_{i,j} - \phi_i\phi_j\right) - \delta^h{}_j\left(\phi_{i,k} - \phi_i\phi_k\right).$$

Contracting this formula with respect to h, k, we obtain the following relation of the Ricci curvatures of Γ and $\bar\Gamma$:

$$\bar R_{ij} = R_{ij} + (n-1)\left(\phi_{i,j} - \phi_i\phi_j\right) + \phi_{i,j} - \phi_{j,i}.$$

Projective Weyl tensor.

It is the following tensor field:

$$W^h_{\ ijk} = R^h_{\ ijk} - \tfrac{1}{n-1}\left(\delta^h_{\ k} R_{ij} - \delta^h_{\ j} R_{ik}\right) + \tfrac{1}{n+1}\left(\delta^h_{\ i} R_{[jk]} - \tfrac{1}{n-1}\left(\delta^h_{\ k} R_{[ji]} - \delta^h_{\ j} R_{[ki]}\right)\right).$$

Theorem (Weyl, Schouten). Projective Weyl tensor is a tensor invariant of a projective structure, i.e. it does not depend on the choice of connection within the projective structure.

Proof. Substituting the formulas

$$\bar{R}^h_{\ ijk} = R^h_{\ ijk} + \left(\phi_{j,k} - \phi_{k,j}\right)\delta^h_{\ i} + \delta^h_{\ k}\left(\phi_{i,j} - \phi_i\phi_j\right) - \delta^h_{\ j}\left(\phi_{i,k} - \phi_i\phi_k\right).$$

$$\bar{R}_{ij} = R_{ij} + (n-1)\left(\phi_{i,j} - \phi_i\phi_j\right) + \phi_{i,j} - \phi_{j,i}.$$

in the definition of the projective Weyl tensor changing the covariant derivative in Γ by the covariant derivative in $\bar{\Gamma}$, we see after an hour of calculations that all ϕ's disappear.

Liouville invariant

In dimension 2, Weyl tensor is necessary identically zero, since each $(1,3)$ tensor with its symmetries is zero. Fortunately and exceptionally, there is one more tensor invariant in dimension 2:

Theorem (Liouville 1889). The tensor field $L_{ijk} := R_{ij,k} - R_{ik,j}$ is the same for projectively equivalent 2-dimensional metrics.

Proof. At least in the Riemannian case, it is sufficient to prove for metrics from the Dini theorem, and there one can work by direct calculation.

$$\bar{R}_{ij} = R_{ij} + (n-1)\left(\phi_{i,j} - \phi_i\phi_j\right) + \phi_{i,j} - \phi_{j,i}.$$

in the definition of L and changing the covariant derivative in Γ by the covariant derivative in $\bar{\Gamma}$ we again see that all terms containing ϕ disappear (assuming $n = 2$). □

Remark. There is a similar story in conformal geometry: conformal Weyl tensor vansihes for $n \leq 3$ but in dimension 3 there exists an additional conformal invariant and in dimension 2 conformal geometry is not interesting all. There is a deep explanation of this similarity and there are many results in $n+1$ dimensional conformal geometry that are visually similar to results in n-dimensional projective geometry; we will not discuss in in this lecture course but just remember that many ideas from my course can be effectively used in the conformal geometry as well.

How many essential components does L_{ijk} have and when it vanishes?

Theorem (Liouville 1889). The tensor field $L_{ijk} := R_{ij,k} - R_{ik,j}$ is a tensor invariant in dim 2.

The tensor L_{ijk} is skew-symmetric in j, k, assuming $n = \dim M = 2$ it implies that it has two essential components and can be written in the form $L = (L_1 dx^1 + L_2 dx^2) \otimes (dx^1 \wedge dx^2)$.

Theorem. Let $\nabla^g = (\Gamma^i_{jk})$ be the Levi-Civita connection of g on 2-dim M. Then, $L_{ijk} \equiv 0$ if and only if g has constant curvature.

Proof. It is well-known (and follow from the symmetries of the curvature tensor) that the 2-dim manifold are automatic Einstein in the sense that

$$R_{ij} = \tfrac{1}{2} R g_{ij}.$$

Calculating L_{ijk} gives

$$L_{ijk} = R_{ij,k} - R_{ij,k} = \tfrac{1}{2}\left(R_{,k} g_{ij} - R_{,j} g_{ik}\right).$$

Since g is nondegenerate, vanishing of L implies vanishing of dR and hence the constancy of the curvature. □

Remark. We also see (or can easily check) that $L_{ijk} = dR \otimes (dx \wedge dy)$.

$W \equiv 0$ implies constant curvature

Theorem. Let Γ be the Levi-Civita connection of g on M with $n > 2$. Then, $W^h{}_{ijk} \equiv 0$ if and only if g has constant sectional curvature.

Proof. For Levi-Civita connections the Ricci tensor is symmetric so the formula for W reads

$$W^h{}_{ijk} = R^h{}_{ijk} - \tfrac{1}{n-1}\left(\delta^h{}_k R_{ij} - \delta^h{}_j R_{ik}\right).$$

If $W \equiv 0$, we obtain

$$R^h{}_{ijk} = \tfrac{1}{n-1}\left(\delta^h{}_k R_{ij} - \delta^h{}_j R_{ik}\right).$$

After lowing the index we have therefore

$$R_{hijk} = \tfrac{1}{n-1}\left(g_{hk} R_{ij} - g_{hj} R_{ik}\right).$$

We see that the left-hand-side is symmetric with respect to $(h, i, j, k) \longleftrightarrow (j, k, h, i)$, so should be the right-hand-side, which implies that R_{ij} is proportional to g_{ij}, $R_{ij} = \frac{R}{n} g_{ij}$ so we have

$$R_{hijk} = \tfrac{R}{n(n-1)}\left(g_{hk} g_{ij} - g_{hj} g_{ik}\right)$$

which is equivalent to "sectional curvature is constant". □

By-product: Beltrami Theorem

Theorem. Let $\nabla^g = (\Gamma^i_{jk})$ be the Levi-Civita connection of g on 2-dim M. Then, $L_{ijk} \equiv 0$ if and only if g has constant curvature.

Theorem. Let Γ be the Levi-Civita connection of g on M with $n > 2$. Then, $W^h_{ijk} \equiv 0$ if and only if g has constant sectional curvature.

Corollary (Beltrami Theorem; Beltrami 1865 for dim 2; Schur 1886 for dim> 2). A metric projectively equivalent to a metric of constant curvature has constant curvature.

Nice result for today: projective Lichnerowicz conjecture

Theorem. Let (M, g) be a compact Riemannian manifold such that the sectional curvature is not constant positive. Then, any projective vector field is a Killing vector field.

Remark. We have seen in Lecture 3 that the algebra of projective vector fields of the round sphere is $sl(n+1)$ and is bigger than the algebra of isometries which is $so(n+1)$.

Remark. We have also seen that in dimension 2 there are (local) metrics of nonconstant curvature admitting projective vector fields. One can construct similar examples in all dimensions. Theorem above says that these examples can not be extended to a closed manifold.

Was a very popular conjecture

Special cases were proved before by French, Japanese and Soviet geometry schools.

France (Lichnerowicz)	Japan (Yano, Obata, Tanno)	Soviet Union (Raschewskii)
Couty (1961) proved the conjecture assuming that g is Einstein or Kähler	Yamauchi (1974) proved the conjecture assuming that the scalar curvature is constant	Solodovnikov (1956) proved the conjecture assuming that all objects are real analytic and that $n \geq 3$.

Remark. Stronger statements are also true:

- The statement remains true if one replaces "closed" by "complete", assumes in addition that the projective vector field is complete, and also allows flat metrics:
- **Theorem (M∼ 2003).** *On a complete Riemannian manifold such that its curvature is not nonnegative constant* $Proj_0 = Iso_0$.
- **Theorem (M∼ 2016)** On complete manifolds $|Proj/Iso| \leq 2$ (Zeghib 2013 has shown $|Proj/Iso| \leq 2n$ for closed manifolds).
- The statements also remains true in the Lorentzian siganture (Bolsinov-Rosemann-Matveev 2015).

A difficulty of dimensions $n \geq 3$ which I avoid by additional assumption.
In dimension 2, in the solution of the 1st Lie problem, we assumed w.l.o.g. that $\dim(Sol([\Gamma]) = 2$.

The argument was: there exists a 2-dimensional invariant subspace of $Sol([\Gamma]$ and if the solutions from this subspace are degenerate there exists a Killing vector field.

The latter arguments does not work in dimensions ≥ 3, but still we may assume that $\dim(Sol([\Gamma]) \leq 2$ because of the following nontrivial theorem whose proof involves new circle of ideas and will be explained at the end of this lecture.

Theorem (M$\sim$ 2003). On a closed Riemannnian manifold such that its sectional curvature is not constant positive, $\dim\left(Sol([\Gamma])\right) \leq 2$.

Plan of the proof of the Lichnerowicz conjecture.

Setup.

- Our manifold is closed and Riemannian.
- The projective structure of the metric admits a projective vector field.
- We assume that $\dim\left(Sol([\Gamma])\right) \leq 2$.
- Our goal is to show that this vector field is a Killing vector field unless g has constant sectional curvature

The case $2 > \dim(Sol([\Gamma]) \quad = 1$

If $\dim\left(Sol([\Gamma])\right) = 1$, every two projective related metrics are proportional. Then, a projective vector field v is a homothety vector field. Since our manifold is closed, every homothety is isometry so our vector field is a Killing vector field as we want.

The case $\dim(Sol([\Gamma]) = 2$

Important observation already used in the solution of Lie problems. $L_v : Sol([\Gamma]) \to Sol([\Gamma])$, where L_v is the Lie derivative.

After appropriate choice of a basis in $Sol([\Gamma])$, we obtained that the v-Lie derivative $\sigma, \bar\sigma$ are given by

$$\begin{bmatrix} L_v\sigma & = \lambda\sigma & \\ L_v\bar\sigma & = & \mu\bar\sigma \end{bmatrix} \quad \begin{bmatrix} L_v\sigma & = \lambda\sigma & +\mu\bar\sigma \\ L_v\bar\sigma & = -\mu\sigma & +\lambda\bar\sigma \end{bmatrix} \quad \begin{bmatrix} L_v\sigma & = \lambda\sigma & +\bar\sigma \\ L_v\bar\sigma & = & \lambda\bar\sigma \end{bmatrix}.$$

Thus, the evolution of the solutions along the flow Φ_t of v is

$$\begin{bmatrix} \Phi_t^*\sigma & = e^{\lambda t}\sigma & \\ \Phi_t^*\bar\sigma & = & e^{\mu t}\bar\sigma \end{bmatrix}$$

$$\begin{bmatrix} \Phi_t^*\sigma & = e^{\lambda t}\cos(\mu t)\sigma & +e^{\lambda t}\sin(\mu t)\bar\sigma \\ \Phi_t^*\bar\sigma & = -e^{\lambda t}\sin(\mu t)\sigma & +e^{\lambda t}\cos(\mu t)\bar\sigma \end{bmatrix}$$

$$\begin{bmatrix} \Phi_t^*\sigma & = e^{\lambda t}\sigma & +te^{\lambda t}\bar\sigma \\ \Phi_t^*\bar\sigma & = & e^{\lambda t}\bar\sigma \end{bmatrix}.$$

We will consider all these three cases separately.

The simplest case is when the evolution is given by

$$\begin{bmatrix} \Phi_t^*\sigma & = e^{\lambda t}\cos(\mu t)\sigma & + e^{\lambda t}\sin(\mu t)\bar{\sigma} \\ \Phi_t^*\bar{\sigma} & = -e^{\lambda t}\sin(\mu t)\sigma & + e^{\lambda t}\cos(\mu t)\bar{\sigma} \end{bmatrix}.$$

Suppose our metric corresponds to the element $a\sigma + b\bar{\sigma}$.
Its evolution is given by

$$\begin{aligned} \Phi_t^*(a\sigma + b\bar{\sigma}) &= a(e^{\lambda t}\cos(\mu t)\sigma + e^{\lambda t}\sin(\mu t)\bar{\sigma}) \\ &\quad + b(-e^{\lambda t}\sin(\mu t)\sigma + e^{\lambda t}\cos(\mu t)\bar{\sigma}) \\ &= e^{\lambda t}\sqrt{a^2+b^2}(\cos(\mu t + \alpha)\sigma + \sin(\mu t + \alpha)\bar{\sigma}), \end{aligned}$$

where $\alpha = \arccos(a/(\sqrt{a^2+b^2}))$.

Now, we use that the metric is Riemannian. Then, for any point x there exists a basis in T_xM such that σ and $\bar{\sigma}$ are given by diagonal matrices: $\sigma = diag(s_1, s_2, \ldots)$ and $\bar{\sigma} = diag(\bar{s}_1, \bar{s}_2, \ldots)$.

Then, $\Phi_t^*(a\sigma + b\bar{\sigma})$ at this point is also diagonal with the ith element $e^{\lambda t}\sqrt{a^2+b^2}(\cos(\mu t + \alpha)s_i + \sin(\mu t + \alpha)\bar{s}_i)$.

Clearly, for a certain t we have that $\Phi_t^*(a\sigma + b\bar{\sigma})$ is degenerate which contradicts the assumption, □

The proof is is similar when the evolution is given by

$$\begin{bmatrix} \Phi_t^*\sigma & = e^{\lambda t}\sigma & + te^{\lambda t}\bar{\sigma} \\ \Phi_t^*\bar{\sigma} & = & e^{\lambda t}\bar{\sigma} \end{bmatrix}.$$

We again suppose that our metric corresponds to the element $a\sigma + b\bar{\sigma}$.

Its evolution is given by

$$\begin{aligned} \Phi_t^*(a\sigma + b\bar{\sigma}) &= a(e^{\lambda t}\sigma + e^{\lambda t}t\bar{\sigma}) + b(e^{\lambda t}\bar{\sigma}) \\ &= e^{\lambda t}(a\sigma + (b + at)\bar{\sigma}). \end{aligned}$$

We again see that unless $a \neq 0$ there exists t such that $\Phi_t^*(a\sigma + b\bar{\sigma})$ is degenerate which contradicts the assumption.

Now, if $a = 0$, then g corresponds to $\bar{\sigma}$ and v is its Killing vector field, □

The most complicated case is when the evolution is given by the matrix

$$\begin{bmatrix} \Phi_t^*\sigma & = e^{\lambda t}\sigma & \\ \Phi_t^*\bar\sigma & = & e^{\mu t}\bar\sigma \end{bmatrix}. \tag{2}$$

The case $\lambda = \mu$ is trivial, in this case the projective vector field is homothety vector field. We assume $\lambda > \mu$.

We may assume that g corresponds to the solution $\sigma + \bar\sigma$.
Consider, for each $t \in \mathbb{R}$, the $(1,1)$-tensor

$$A_t = (\sigma + \bar\sigma)^{-1}\Phi_t^*(\sigma + \bar\sigma) = (\sigma + \bar\sigma)^{-1}(e^{\lambda t}\sigma + e^{\mu t}\bar\sigma).$$

Take a point p and consider a basis such that

$$g = \mathrm{diag}(1, ..., 1), \quad \sigma = \mathrm{diag}(s_1, ..., s_n), \bar\sigma = \mathrm{diag}(\bar s_1, ..., \bar s_n)$$

(Since $\sigma + \bar\sigma$ corresponds to g, we have $\bar s_i = 1 - s_i$).
In this basis, we have

$$A_t = \mathrm{diag}(s_1 e^{\lambda t} + \bar s_1 e^{\mu t}, ...).$$

Next, for each $t \in \mathbb{R}$, consider the tensor

$$G_t = g^{-1}\Phi_t^* g.$$

Because of the relation $g^{-1} = \sigma \, |\det(\sigma)|$ (see Lecture 2), we have

$$G_t = \mathrm{diag}\left(\frac{1}{(s_1 e^{\lambda t} + \bar s_1 e^{\mu t})\prod_i (s_i e^{\lambda t} + \bar s_i e^{\mu t})}, ...\right).$$

$$G_t = \operatorname{diag}\left(\frac{1}{(s_1 e^{\lambda t}+\bar{s}_1 e^{\mu t})\prod_i (s_i e^{\lambda t}+\bar{s}_i e^{\mu t})}, \dots\right).$$

Let us assume for simplicity that all $s_i, \bar{s}_i \neq 0$. Since $\lambda > \mu$,

$$G_t \overset{t\to+\infty}{\longrightarrow} \operatorname{diag}(e^{-(n+1)\lambda t}, \dots) \text{ and } G_t \overset{t\to-\infty}{\longrightarrow} \operatorname{diag}(e^{(n+1)\mu t}, \dots). \qquad (\star)$$

Consider now the function $f = (|W|_g)^2 = W^i{}_{jk\ell} g_{ii'} g^{jj'} g^{kk'} g^{\ell\ell'} W^{i'}{}_{j'k'\ell'}$. It is a smooth function on the manifold. At points such that $W \neq 0$ we have $f(p) \neq 0$.

Since $\Phi_t^*(g) = gG_t$ and because of $(\star)$ we have that $\Phi_t^*(g)$ has asymptotic $e^{-2(n+1)\lambda t}$ for $t \to +\infty$ and $e^{-2(n+1)\mu t}$ for $t \to -\infty$.

Now, BECAUSE W IS PROJECTIVELY INVARIANT, $\Phi_t^* W = W$. Thus, for $t \to \infty$,

$$f(\Phi_t(p)) = \Phi_t^* f(p) = |\Phi_t^* W|^2_{\Phi_t^* g} \sim \text{const}\, e^{2(n+1)\lambda t}$$

(where const $= 0$ iff $W = 0$)

and for $t \to -\infty$ we have $f(\Phi_t(p)) \sim \text{const}\, e^{-2(n+1)\mu t}$.

$$f(\Phi_t(p)) = \Phi_t^* f(p) = |\Phi_t^* W|^2_{\Phi_t^* g} \sim \text{const}\, e^{2(n+1)\lambda t}$$

(where const $= 0$ iff $W = 0$) and for $t \to -\infty$ we have

$$f(\Phi_t(p)) \sim \text{const}\, e^{-2(n+1)\mu t}$$

We see that if $W(p) \neq 0$ then the smooth function f on a compact manifold is unbounded, which gives a contradiction. □

Remark. We had an additional assumption: all $s_i \neq 0$. It is not essential, one simply should be slightly more careful.

Remark. In the 2 dim case one should replace W by the Liouville invariant L_{ijk}.

Summary of the proof of the projective Lichnerowicz conjecture

Theorem (Lichnerowicz conjecture). Let (M, g) be a compact Riemannian manifold such that the sectional curvature is not constant positive. Then, any projective vector field is a Killing vector field.

- We assumed in addition that $\dim(Sol([\Gamma])) = 2$ and justified this assumption by certain fact we did not prove.
- Then, we used the invariance of the metrization equation and obtained that the evolution of the solutions along the flow of the projective vector field is given by one of the three cases:

$$\left[\begin{array}{lll} \Phi_t^*\sigma & = e^{\lambda t}\sigma & \\ \Phi_t^*\bar\sigma & = & e^{\mu t}\bar\sigma \end{array}\right], \left[\begin{array}{lll} \Phi_t^*\sigma & = e^{\lambda t}\cos(\mu t)\sigma & +e^{\lambda t}\sin(\mu t)\bar\sigma \\ \Phi_t^*\bar\sigma & = -e^{\lambda t}\sin(\mu t)\sigma & +e^{\lambda t}\cos(\mu t)\bar\sigma \end{array}\right], \left[\begin{array}{lll} \Phi_t^*\sigma & = e^{\lambda t}\sigma & +te^{\lambda t}\bar\sigma \\ \Phi_t^*\bar\sigma & = & e^{\lambda t}\bar\sigma \end{array}\right].$$

- In all three cases some geometrically constructed (and therefore continuous) function is unbounded which can not happen on a closed manifold: in the blue and **black** cases it $\frac{\det(g)}{\det(\Phi_1^* g)}$. In the red case the function is $f = (|W|_g)^2 = W^i{}_{jk\ell} g_{ii'} g^{jj'} g^{kk'} g^{\ell\ell'} W^{i'}{}_{j'k'\ell'}$. It is unbounded unless $W \equiv 0$. In the proof we have used that W is projectively invariant, and that $W = 0$ implies constant curvature.

A bit of philosophy

Felix Klein, 1873, Vergleichende Betrachtungen über neuere geometrische Forschungen

Problem I: *Es ist eine Mannigfaltigkeit und in derselben eine Transformationsgruppe gegeben. Man entwickle die auf die Gruppe bezugliche Invariantentheorie.*

English Translation. Given a manifoldness and a group of transformations of the same; to develop the theory of invariants relating to that group.

In our case we had a manifold, a group of projective transformations, a projective invariants for them, and used them to prove Lichnerowicz conjecture.

In the previous lectures we also had projectively invariants objects, and they were extremely effective.

Why dim(Sol)$\leq$ 3

Theorem (M$\sim$ 2003). On a complete Riemannnian manifold such that its sectional curvature is not constant positive, $\dim(Sol([\Gamma])) \leq 2$.
I am going to explain the circle of ideas leading to the proof of this Theorem:

Remark. Theorem survives for any signature (Kiosak-Matveev 2009) and also for closed noncomplete manifolds (Matveev-Mounoud 2012).

Main message for you: linear algebra is important: solving problems in local differential geometry try to get as much info as it is possibly from linear algeba

How to come to a linear algebraic problem: Ricci compatibility conditions:

Folklore forgotten by most modern mathematicians: Ricci compatibility conditions:
For any $(2,0)$-tensor a^{ij} we have:

$$a^{ij}{}_{,km} - a^{ij}{}_{,mk} = R^i{}_{skm}a^{sj} + R^j{}_{skm}a^{is}. \qquad (R)$$

Proof: For (1,0)-tensors the analogous formula reads

$$a^i{}_{,km} - a^i{}_{,mk} = R^i{}_{skm}a^s$$

and is a definition of the curvature tensor. In order to get (R), just decompose a^{ij} in the sum of the tensor products of vectors

$$a^{ij}{}_{,km} - a^{ij}{}_{,mk} = R^i{}_{skm} a^{sj} + R^j{}_{skm} a^{is}. \qquad (R)$$

Let $\bar{\sigma}$ be a solution of the metrization equation for the Levi-Civita connection of the metric g:

$$\sigma^{ij}{}_{,k} = \delta^i_k \mu^j + \delta^j_k \mu^i.$$

Now, tensor-multiply it by $Vol_g^{-\frac{2}{n+1}}$: using that the volume form is parallel for the Levi-Civita connection we obtain the equation for symmetric tensor a^{ij} and vectorfield λ^i:

$$a^{ij}{}_{,k} = \delta^i_k \lambda^j + \delta^j_k \lambda^i.$$

Remark. This equations will be important later, we we call them metrization equations in the presence of metric.

Now differentiate this equation and plug in (R): we will obtain:

$$a^{ip} R^j{}_{pk\ell} + a^{pj} R^i{}_{pk\ell} = \lambda^i{}_{,\ell} \delta^j_k + \lambda^j{}_{,\ell} \delta^i_k - \lambda^i{}_{,k} \delta^j_\ell - \lambda^j{}_{,k} \delta^i_\ell.$$

$$a^{ip} R^j{}_{pk\ell} + a^{pj} R^i{}_{pk\ell} = \lambda^i{}_{,\ell} \delta^j_k + \lambda^j{}_{,\ell} \delta^i_k - \lambda^i{}_{,k} \delta^j_\ell - \lambda^j{}_{,k} \delta^i_\ell. \quad (R1)$$

If σ and therefore a^{ij} came from the metric g, then $\lambda^i = 0$ and the equations are just the symmetries of the curvature tensor.

In our assumption, besides this "trivial" a^{ij} we have two other a's, we denote them a^{ij} and $\bar{a}^{ij}$. We have that g^{ij}, a^{ij} and $\bar{a}^{ij}$ are linearly independent.

Linear algebraic Lemma. If $(R1)$ is satisfied for $(a^{ij}, \lambda^i{}_{,j})$ and $(\bar{a}^{ij}, \bar{\lambda}^i{}_{,j})$ such that g^{ij}, a^{ij} and $\bar{a}^{ij}$ are linearly independent, then for some B we have

$$\lambda^i{}_{,j} = \rho \delta^i_j + B a^i{}_j, \quad \bar{\lambda}^i{}_{,j} = \bar{\rho} \delta^i_j + B \bar{a}^i{}_j.$$

Remark. Linear algebra is quite nontrivial; it uses for example the symmetries of the Riemann curvature tensor

$$\lambda^i{}_{,j} = \rho\delta^i_j + Ba^i{}_j, \quad \bar\lambda^i{}_{,j} = \bar\rho\delta^i_j + B\bar a^i{}_j \quad (R3)$$

Lemma. If (a^{ij}, λ^i), $(\bar a^{ij}, \bar\lambda^i)$ are solutions of the metrization equation in the presence of metric such that g^{ij}, a^{ij} and $\bar a^{ij}$ are linear independent. Assume in additions that (R3) is fulfilled for a function B. Then, this function B is a constant.

Idea of the Proof. Differentiate (R3) and put it in the equation

$$\lambda^i{}_{,km} - \lambda^i{}_{,mk} = R^i{}_{skm}\lambda^s.$$

You will have a system of linear equations on $B_{,i}$ and after nontrivial linear algebra one obtains that $B_{,i} = 0$.

Remark. As a byproduct you also obtain that $\rho_{,i} = 2B\lambda_i$.

Remark. Assuming $B \neq 0$, one can rescale the metric to make $B = -1$

Conification

Thus, we arrived to the following system of equaitons:

$$\begin{array}{rcl} a^{ij}{}_{,k} & = & \delta^i_k\lambda^j + \delta^j_k\lambda^i \\ \lambda^i{}_{,j} & = & \rho\delta^i_j - a^i{}_j \\ \rho_{,i} & = & -2\lambda_i \end{array}$$

This system of equations is closely related to the following geometric situation:

Fact. Consider the cone over (M, g): the manifolds is $\mathbb{R}_{>0} \times M$ and the metric is $dt^2 + t^2 g$. Then, the above equations are just the equations for the following $(2,0)$-tensor on the cone to be parallel:

$$\begin{pmatrix} \rho & \lambda^1 & \cdots & \lambda^n \\ \lambda^1 & & & \\ \vdots & & a^{ij} & \\ \lambda^n & & & \end{pmatrix}$$

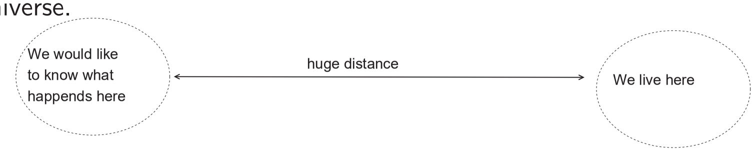
But parallel tensors on the cone of a closed manifold do not exist!!!

Then, the following theorem whose most complicated case (when g is Riemannian) was done independently by Gallot and Tanno 1979 — and is closely related to some obscurely published result of Solodovnikov 1956; most trivial case (when $-g$ is Riemannian) was done by Yano and Hiramata 1982 and the remaining case is in Matveev-Mounoud 2012, finishes the proof.

Theorem. Cone over closed manifold different from the round sphere does not admit parallel $(2, 0)$ tensors.

Projective structures coming from relativity

Suppose we would like to understand the structure of the space-time (i.e., a 4-dimensional metric of Lorenz signature) in a certain part of the universe.



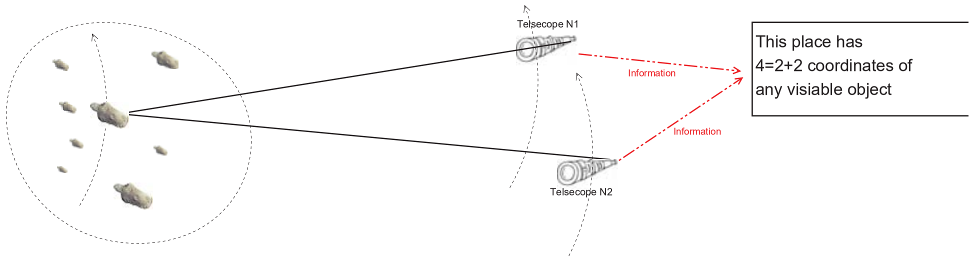
We assume that this part is far enough so the we can use only telescopes (in particular we can not send a space ship there).

We still assume that the telescopes can see sufficiently many objects in this part of universe.

Then, if the relativistic effects are not negligible (that happens for example is the objects in this part of space time are sufficiently fast or if this region of the universe is big enough),
we obtain as a rule the world lines of the objects as unparameterized curves.

In many cases, we do can get unparameterized geodesics with the help of astronomic observations

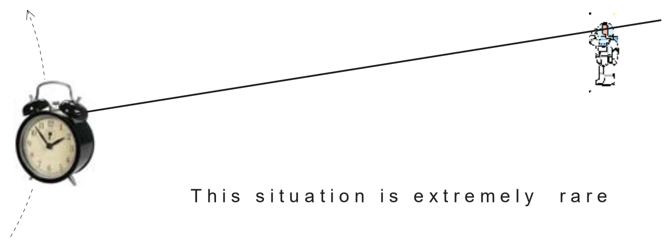
One can obtain unparameterized geodesics by observation:



We take 2 freely falling observers that
measure two angular coordinates of the visible objects
and send this information to one place. This place will have
4 functions angle(t) for every visible object
which are in the generic case 4 coordinates of the object.

In many cases, the only thing one can get by observations are unparametrized geodesics

If one can not register a periodic process on the observed body, one can not get the own time of the body



This situation is extremely rare

Problem 1. How to reconstruct a metric by its unparameterized geodesics?

The mathematical setting: *We are given a family of smooth curves $\gamma(t;\alpha)$ in $U \subseteq \mathbb{R}^4$; we assume that the family is sufficiently big in the sense that $\forall x_0 \in U$ we have enough for our goals curves coming through this point. We need to reconstruct the metric whose geodesics are these curves.*

In the realm of general relativity, the problem was explicitly stated by

Jürgen Ehlers 1972, who said that "We reject clocks as basic tools for setting up the space-time geometry and propose ... freely falling particles instead. We wish to show how the full space-time geometry can be synthesized ... . Not only the measurement of length but also that of time then appears as a derived operation."

In my first lectures we learned (in dimension 2, but the ideas are the same) the first steps to solve the problem! We know how to reconstruct a projective class, what PDE should we solve!

If the searched metric is Ricci-flat, there exists a trick that simplifies the algorithm.

We consider the projective Weyl tensor introcuded at the beginning of the lecture

$$W^i{}_{jk\ell} := R^i{}_{jk\ell} - \tfrac{1}{n-1}\left(\delta^i_\ell R_{jk} - \delta^i_k R_{j\ell}\right) \tag{3}$$

Now, from the formula (3), we know that, if the searched $\bar{g}$ is Ricci-flat, projective Weyl tensor coincides with the Riemann tensor $\bar{R}^i{}_{jk\ell}$ of $\bar{g}$. Thus, if we know the projective class of the Ricci-flat metric $\bar{g}$, we know its Riemann tensor. Then, the metric $\bar{g}$ must satisfy the following system of equations due to the symmetries of the Riemann tensor:

$$\begin{cases} \bar{g}_{ia} W^a{}_{jkm} + \bar{g}_{ja} W^a{}_{ikm} = 0 \\ \bar{g}_{ia} W^a{}_{jkm} - \bar{g}_{ka} W^a{}_{mij} = 0 \end{cases} \tag{4}$$

The first portion of the equations is due to the symmetry $(\bar{R}_{ijkm} = -\bar{R}_{jikm})$, and the second portion is due to the symmetry $(\bar{R}_{kmij} = \bar{R}_{ijkm})$ of the curvature tensor of $\bar{g}$.

$$\begin{cases} \bar{g}_{ia} W^a{}_{jkm} + \bar{g}_{ja} W^a{}_{ikm} = 0 \\ \bar{g}_{ia} W^a{}_{jkm} - \bar{g}_{ka} W^a{}_{mij} = 0 \end{cases}$$

The system (4) is a system of linear equations on $\bar{g}(x_0)_{ij}$. The number of equations (around 100) is much bigger than the number of unknowns (which is 10). It is expected therefore, that a generic projective Weyl tensor $W^i{}_{jkl}$ admits no more than one-dimensional space of solutions (by assumtions, our W admits at least one-dimensional space of solutions). The expectation is true, as the following classical result shows

Theorem (Folklore – Petrov, Hall, Rendall, Mcintosh) Let $W^i{}_{jk\ell}$ be a tensor in $\mathbb{R}^4$ such that it is skew-symmetric with respect to k, ℓ and such that its traces $W^a{}_{ak\ell}$ and $W^a{}_{ja\ell}$ vanish. Assume that for all 1-forms $\xi_i \neq 0$ we have $W^a{}_{jk\ell}\xi_a \neq 0$. Then, the equations (4) have no more than one-dimensional space of solutions.

The case when $W^a{}_{jk\ell}\xi_a = 0$ for some ξ^α is quite exotic – it can not exist in the Riemannian case; it actually implies that ξ_a is light like and in our situation this case can easily be finished by other methods.