Exercises to lectures of Matveev

The exercises that will be important for the next lectures are marked by (!); complicated exercises are marked by *, solving them is a big achievement. The tutors of the exercise session also see these exercises first time and possibly did not think in this direction before, insist that they explain you also complicated exercise and enjoy seeing high level mathematicians trying to solve complicated exercises from scratch.

- Projective structure.
  1. Prove Theorem 1: $\nabla = (\Gamma^i_{jk})$ is projectively equivalent to $\bar\nabla = (\Gamma^i_{jk})$, if and only if there exists an 1-form $\phi = \phi_i$ such that
  $$\bar\Gamma^i_{jk} = \Gamma^i_{jk} + \phi_k \delta^i_j + \phi_j \delta^i_k.$$
  2. Prove Corollary on page 7: show that the kernel of the (linear) mapping $\Gamma^i_{jk} \mapsto (K_0, K_1, K_2, K_3)$ defined in (1) consists of tensors $T^i_{jk} = \phi_k \delta^i_j + \phi_j \delta^i_k$.
  3. construct a sufficiently big smooth family of smooth curves such that it is sufficiently big, i.e., in each point in each direction there exists a curve through this point in this direction, but such that it is not a projective structure.

     Caution: be careful with curves tangent to the vertical direction – the family may fail to be smooth at these curves

- Projective weights and projectively invariant operators on weighted vectorfields
  1. Write in indices how $(1,0)$-tensors of projective weight 1 transform under the coordinate change. Write in indices the formula for the covariant derivative of $(1,0)$-tensors of projective weight 1. Use **Way B** for coordinate representation.
  2. (!) Prove that for $(1,0)$-tensors of projective weight 1 the operation
  $$\sigma \mapsto Trace\_Free\_Part\_Of\nabla(\sigma) = \sigma^i{}_{,j} - \frac{1}{n}\sigma^s{}_{,s}\delta^i_j.$$
  is projectively invariant: it does not depend on the choice of the affine connection in the projective class.
  3. (!) Show that a connection $\nabla = (\Gamma^i_{jk})$ is has symmetric Ricci tensor iff locally there exists a parallel volume form. Moreover, such volume form is unique up to multiplication by a constant. Hint: preferable direct calculations but if you hate them use that for torsionfree connection we have $[U, V] = \nabla_U V - \nabla_V U$.
  4. * Show by example that globally, on nonsimply connected manifolds, the existence of such a parallel volume form may fail. Hint: Look for example in the class on projectively flat manifolds. Key words are "Hopf foliation".

- Using concervative quantities.

1. (!) Prove: if two 2-dimensional metrics at least one of them is complete are projectively equivalent and proportional in more than 4 points than they are proportional everywhere with a constant coefficient.

   Hint: first show that the coefficient of proportionality should necessary be a constant.

2. Prove the Weyl Theorem 1924: it two metrics of any dimension ≥ 2 are projectively equivalent everywhere and conformally equivalent on an open nonempty subset then they are proportional with a constant coefficient.

   Hint: Act as in exercise above. Replace the argument that intersection of different quadrics contains no more than 4 points by the following: intersections of quadrics is an algebraic subset of and any algebraic subset of $\mathbb{R}^n$ containing an open nonnempty subset is the whole $\mathbb{R}^n$.

3. * Prove that in all dimensions ≥ 2 the set of the points where the metrics are proportional is a totally geodesic submanifold.

4. * Prove (at least in dimension 2) that a generic in the C^2-sense metric does not admit nonproportional projective equivalence

   Hint. The general ideas is the same as everywhere above; you need to consider conservative quantities. If brainstorm does not work, read §2.1 of the paper of Kruglikov and Matveev "The geodesic flow of a generic metric does not admit nontrivial integrals polynomial in momenta".

- Dini Theorem and Lie problem on the torus

  1. Show that for two projectively equivalent 2-dimensional metrics near the points where they are not proporional one can canonically construct two commutative eigenvectorfields such that in the corresponding coordinate system the metrics are given by Dini's formulas.

     Hint: observe that in the Dini coordinate system the functions $X(x)$ and $Y(y)$ can be defined in the terms of eigenvalues of $g^{-1}\bar{g}$; and then use them to "normalise" the eigenvectors.

  2. Use this to describe all pairs of projectively equivalent metrics on the torus Hint: In the exercise secton yesterday we have shown that on complete manifold the number of the points where the metrics are proportional is at most two. Use this to show that on the torus projectively equivalent metrics are proportional at no points.

  3. Show that on the 2- torus equipped with a nonflat Riemannian metric any projective vector field is Killing (as will be explained tomorrow, this is a special case of the Lichnerowicz conjecture).

     Hint: We know the formulas for the metrics admitting projective vector field; see that solution of the Lie problem. You need to show that in view of the previous exercise the formulas are defined on the whole torus and then the evolution of functions $X(x)$ and $Y(y)$ leads to a contradiction.

- Lie algebra of projective vector fields

  1. Show that in dimension 2 and assuming that the metric is not of constant curvature situation any projective vector field V and any Killing vector field K satisfy the relation $[K, V] = \text{const} \cdot K$.

     Hint: Use that there is only one, up to a constant coefficient, Killing vector field and apply the PDE-trick from the lecture.

2. (*) Show without using theorem about the solution of the 2nd problem of Lie that const from $[K, V] = \text{const} \cdot K$ can not be zero for metrics of nonconstant curvature

- Generalize the Lagrange example for metrics of constand negative curvature (show that they are projectively equivalent to a flat metric)