Systolic and other inequalities

This is a survey course on a collection of geometric inequalities (some true and proved, some false with counterexamples, and some that are still open questions). The inequalities generally involve lower bounds on the volume of a Riemannian manifold in terms of other geometric quantities (usually without curvature assumptions). A common thread in these inequalities is that they were each advanced in the seminal and wide ranging paper "Filling Riemannian manifolds" [Gr83].

The course will begin with the isosystolic inequalities of Loewner, Pu, and Bavard giving the optimal systolic ratios for metrics on the two dimensional torus, the projective plane, and the Klein bottle. Here the systole is the length of the shortest non-homotopically trivial closed curve and the systolic ratio of a Riemannian manifold is $\frac{VOL(M)}{systole(M)^n}$. We will also discuss the current state of knowledge of isosystolic ratios for other surfaces. We will then discuss some of the important results and concepts (e.g. the filling radius and the filling volume) in [Gr83]. This will lead us in a number of directions. I mention some below:

- Systolic inequalities in higher dimensions: We will discuss Gromov's systolic inequality for essential manifolds along with some other estimates.
- Systolic freedom: We will discuss some possible systolic-type inequalities that turn out not to hold.
- The boundary rigidity problem and its relation to optimal fillings: We will discuss how the volume (and the metric itself) of a compact Riemannian manifold with boundary is related to the chordal distances (i.e. the length of the shortest path through the manifold) between its boundary points. In particular, we hope to discuss a very new result of Burago-Ivanov.
- Isoembolic inequalities. We discuss relationships between the volume and the injectivity radius of a Riemannian manifold.

The contents of the course is similar to the contents of the survey paper [C-K03] (though with a different emphasis). The attached bibliography contains a number of sources for the course, though the primary sources are [Gr83] and [C-K03]. A more extensive bibliography can be found in the survey [C-K03].

References

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